INFINITE RIGIDITY OF INVERSIVE DISTANCE CIRCLE PACKINGS IN THE POINCARÉ DISK

YANWEN LUO, XU XU, CHAO ZHENG

ABSTRACT. The maximum principle for hyperbolic inversive distance circle packings on polyhedral surfaces is established, which unifies and generalizes existing maximum principles for various types of circle packings in the literature. As an application of this principle, a discrete Schwarz-Ahlfors lemma is established. Furthermore, an infinite rigidity theorem for weighted Delaunay triangulations of the Poincaré disk is proved, which generalizes He's hyperbolic rigidity result [11].

1. INTRODUCTION

1.1. **Background.** Research on circle packing is thriving in the field of discrete geometry, with its origins in Thurston's work on three-dimensional hyperbolic geometry. Thurston [18] introduced Thurston's circle packings with non-obtuse intersection angles, and established their existence and rigidity, known as the famous Koebe-Andreev-Thurston theorem. Later, Ge-Hua-Zhou [5] generalized this theorem to the case of obtuse angles. Inversive distance circle packings, proposed by Bowers-Stephenson [3], are natural generalizations of Thurston's circle packings. Unlike Thurston's circle packings, adjacent circles in inversive distance circle packings can be disjoint, with their relative positions quantified by the inversive distance, a generalization of the intersection angle. Bowers-Stephenson [3] further conjectured that inversive distance circle packings are rigid. For non-negative inversive distances, Guo [8] proved the local rigidity, while Luo [12] established the global rigidity. Subsequently, Xu [19, 20] extended these results to inversive distances greater than -1, thereby completely resolving the Bowers-Stephenson's rigidity conjecture. Most recently, Bobenko-Lutz [1, 2] established the existence of inversive distance circle packings with inversive distances greater than 1. Notably, all the aforementioned works focus on compact surfaces with Euclidean and hyperbolic background geometry, i.e., surfaces with a finite number of vertices. A natural research direction is to generalize these results to non-compact surfaces.

The first result on the infinite rigidity of circle packings was presented by Rodin-Sullivan [16], who proved the infinite rigidity of tangential circle packings with a hexagonal combinatorial structure on the complex plane $\mathbb{C}$. Building on their work, He [9] provided a simplified proof using Schottky groups. Subsequently, Schramm [17] developed a more general combinatorial method, establishing the infinite rigidity of tangential circle packings (without the hexagonal combinatorial constraint) on both the complex plane $\mathbb{C}$ and the Poincaré disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. A more direct proof of Schramm's rigidity result, adopting a similar approach, can be found in [10]. Using network theory from computer science, He [11] extended these rigidity results from tangential circle packings to Thurston's circle packings.

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Inspired by Luo-Sun-Wu's recent work [13] on Luo's vertex scaling, Luo-Xu-Zhang [14] established the infinite rigidity of Euclidean inversive distance circle packings (with inversive distances in $(-\frac{1}{2}, 1]$ or $[0, +\infty)$) on the hexagonal Euclidean plane $\mathbb{C}$, thus generalizing Rodin-Sullivan's classic result (where the inversive distance is 1). In this paper, we establish the infinite rigidity of hyperbolic inversive distance circle packings in the Poincaré disk $\mathbb{D}$, which generalizes He's hyperbolic rigidity result [11].

1.2. **Main results.** Let $(S, \mathcal{T})$ be a triangulated surface (possibly with boundary) with a triangulation $\mathcal{T} = \{V, E, F\}$, where V, E, and F denote the sets of vertices, edges, and faces, respectively. For notation, a vertex, an edge, and a face of $\mathcal{T}$ are denoted by v_i, v_iv_j, and $\triangle v_iv_jv_k$, respectively.

A piecewise hyperbolic metric (PH metric for short) on $(S, \mathcal{T})$ is a function $l : E \to (0, +\infty)$ that induces a non-degenerate hyperbolic triangle on each face $\triangle v_iv_jv_k$ of $\mathcal{T}$, where the edge lengths are l_{ij}, l_{ik}, and l_{jk} (with $l_{ij} = l(v_iv_j)$). For a PH metric $l : E \to (0, +\infty)$ on $(S, \mathcal{T})$, the combinatorial curvature is a map $K : V \to (-\infty, 2\pi)$, which assigns to an interior vertex $v_i \in V$ the value 2π minus the sum of angles of triangles at v_i, and to a boundary vertex $v_i \in V$ the value π minus the sum of angles at v_i.

Definition 1.1. Let $(S, \mathcal{T}, \eta)$ be a weighted triangulated surface with the weight $\eta : E \to (-1, +\infty)$ satisfying $\eta_{ij} = \eta_{ji}$ for all $v_iv_j \in E$. A PH metric $l : E \to (0, +\infty)$ on $(S, \mathcal{T}, \eta)$ is called a hyperbolic inversive distance circle packing metric, if there exists a function $r : V \to (0, +\infty)$ such that for every edge $v_iv_j \in E$, the edge length l_{ij} satisfies

$$\cosh l_{ij} = \cosh r_i \cosh r_j + \eta_{ij} \sinh r_i \sinh r_j. \tag{1}$$

The map $r : V \to (0, +\infty)$ is referred to as a *hyperbolic inversive distance circle packing* on $(S, \mathcal{T}, \eta)$. Thurston's circle packing [18] is a special case of such inversive distance circle packing, corresponding to weights $\eta \in [0, 1]$. Specifically, the weight η_{ij} in (1) denotes the hyperbolic inversive distance between the two hyperbolic circles centered at v_i and v_j with radii r_i and r_j respectively.

For a weight function $\eta : E \to (-1, +\infty)$, we impose the following structure condition:

$$\eta_{ij} + \eta_{jk}\eta_{ik} \geq 0, \quad \eta_{jk} + \eta_{ij}\eta_{ik} \geq 0, \quad \eta_{ik} + \eta_{ij}\eta_{jk} \geq 0 \tag{2}$$

for every triangle $\triangle v_iv_jv_k$. This condition is necessary, as the inversive distance circle packing loses rigidity when the structure condition is omitted. For further details, we refer the reader to [19, 20].

A weight function $\eta : E \to (-1, +\infty)$ on the triangulated surface $(S, \mathcal{T})$ is called *regular* if there exists no pair of triangles $\triangle v_1v_2v_3$ and $\triangle v_1v_2v_4$ satisfying

$$\eta_{12} = 1, \quad \eta_{13} = -\eta_{23}, \quad \eta_{14} = -\eta_{24}.$$

For example, setting $\eta_{12} = 1$ and $\eta_{13} = \eta_{23} = \eta_{14} = \eta_{24} = 0$ yields exactly the exceptional case described in He [11], as illustrated in Figure 1.

Let P_n be an n-sided star-shaped polygon whose boundary vertices $v_1, \dots, v_n$ are cyclically ordered with $v_{n+1} = v_1$. Let v_0 be an interior point of P_n, which induces a triangulation $\mathcal{T}$ of P_n composed of triangles $\triangle v_0v_iv_{i+1}$ for $i = 1, \dots, n$. See Figure 2.

We have the following hyperbolic maximal principle.

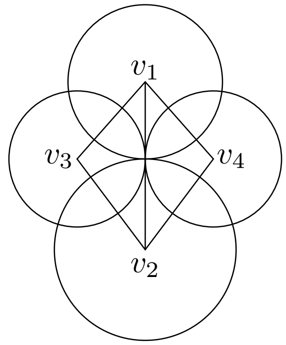

FIGURE 1. The configuration of the four circles.

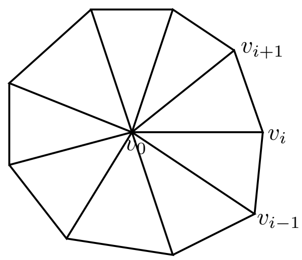

FIGURE 2. A star triangulation of a polygon.

Theorem 1.2. Let η be a regular weight on $(P_n, \mathcal{T})$ satisfying the structure condition (2) with $\eta : E \to (-1, 1]$ or $\eta : E \to [0, +\infty)$. Let r and $\bar{r}$ be two weighted Delaunay hyperbolic inversive distance circle packings on $(P_n, \mathcal{T}, \eta)$, with all corresponding circles contained in the Poincaré disk $\mathbb{D}$. Define $u = \ln \tanh \frac{r}{2}$ and $\bar{u} = \ln \tanh \frac{\bar{r}}{2}$, and let $w_i = \bar{u}_i - u_i$ for each vertex $v_i \in V(\mathcal{T})$. Then the following statements hold:

(i): If the combinatorial curvatures at the interior vertex v_0 satisfy $K_0(r) \geq K_0(\bar{r})$ and $w_0 > 0$, then

$$w_0 < \max_{i\in\{1,2,\dots,n\}} w_i.$$

(ii): If the combinatorial curvatures at the interior vertex v_0 satisfy $K_0(r) \leq K_0(\bar{r})$ and $w_0 < 0$, then

$$w_0 > \min_{i\in\{1,2,\dots,n\}} w_i.$$

Here the weighted Delaunay condition is defined in Subsection 2.3.

Remark 1.3. The case where $\eta : E \to [0, +\infty)$ has been established in [15]. Here, we use a unified approach to prove it. Moreover, Theorem 1.2 generalizes the hyperbolic maximum principle given in Lemma 2.2 of He [11]. Notably, our result does not require P_n to be embedded in the hyperbolic plane $\mathbb{D}$, a condition that is equivalent to $K_0(r) \equiv K_0(\bar{r}) \equiv 0$.

By combining the definition $w = \bar{u} - u$ with the relation between u and r, Theorem 1.2 yields the following discrete Schwarz-Ahlfors lemma directly.

Theorem 1.4 (Discrete Schwarz-Ahlfors lemma)**.** Let η be a regular weight on $(M, \mathcal{T})$ satisfying the structure condition (2) with $\eta : E \to (-1, 1]$ or $\eta : E \to [0, +\infty)$, where $M \subseteq \mathbb{D}$ is a compact set with non-empty boundary. Let r and $\bar{r}$ be two weighted Delaunay hyperbolic inversive distance circle packings on $(M, \mathcal{T}, \eta)$, with all corresponding circles contained in $\mathbb{D}$. Then the following statements hold:

(a): If the combinatorial curvatures $K(r) \geq K(\bar{r})$ for all interior vertices, and $r \geq \bar{r}$ holds for every boundary vertex, then $r \geq \bar{r}$ holds for all vertices.

(b): If the combinatorial curvatures $K(r) \leq K(\bar{r})$ for all interior vertices, and $r \leq \bar{r}$ holds for every boundary vertex, then $r \leq \bar{r}$ holds for all vertices.

We refer the reader to [15] for the rationale for Theorem 1.4 being termed the *Discrete Schwarz-Ahlfors Lemma*.

When the compact subset M in Theorem 1.4 is replaced with the Poincaré disk $\mathbb{D}$, the triangulation $\mathcal{T}$ is required to be an infinite but locally finite triangulation. We have the following infinitely rigidity result.

Theorem 1.5 (Rigidity theorem)**.** Let η be a regular weight on $(\mathbb{D}, \mathcal{T})$ satisfying the structure condition (2) with $\eta : E \to (-1, 1]$ or $\eta : E \to [0, +\infty)$. Let r and $\bar{r}$ be two weighted Delaunay hyperbolic inversive distance circle packings on $(\mathbb{D}, \mathcal{T}, \eta)$, with all corresponding circles contained in $\mathbb{D}$. If the combinatorial curvatures $K(r) \equiv K(\bar{r}) \equiv 0$ for all interior vertices, then $\bar{r} = r$.

Remark 1.6. Theorem 1.5 generalizes He's classical result [11] on the infinite rigidity of Thurston's hyperbolic circle packings, which corresponds to weights $\eta \in [0, 1]$.

1.3. **Organization of the paper.** In Section 2, we first recall the definition of Euclidean inversive distance circle packings. Next, we elaborate the relationship between Euclidean inversive distance and hyperbolic inversive distance. We then review the weighted Delaunay condition in both Euclidean and hyperbolic inversive distance circle packings. Subsequently, we use the Euclidean maximum principle to prove the hyperbolic maximum principle and the discrete Schwarz-Ahlfors lemma. In Section 3, we establish Theorem 1.5.

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2. Hyperbolic maximal principle and discrete Schwarz-Ahlfors lemma

The proof of Theorem 1.2 relies on the Euclidean maximum principle. To this end, we first demonstrate that the PE metric and the PH metric can induce each other. Since a circle in the Poincaré disk can be regarded as both a Euclidean and a hyperbolic circle, we further clarify the relationship between the Euclidean inversive distance and the hyperbolic inversive distance, which are in fact identical. Notably, the weighted Delaunay condition is invariant under the mutual induction between the PE metric and the PH metric. With these premises established, we can then apply the Euclidean maximum principle to prove Theorem 1.2.

2.1. **Euclidean inversive distance circle packings.** A piecewise Euclidean metric (PE metric for short) on $(S, \mathcal{T})$ is a function $L : E \to \mathbb{R}_{>0}$ that induces a non-degenerate Euclidean triangle on each face $\triangle v_i v_j v_k$ of $\mathcal{T}$, where the edge lengths are L_{ij}, L_{ik}, and L_{jk} (with $L_{ij} = L(v_i v_j)$). A PE metric $L : E \to (0, +\infty)$ on $(S, \mathcal{T}, \eta)$ is called a Euclidean inversive distance circle packing metric, if there exists a function $R : V \to (0, +\infty)$ such that for every edge $v_i v_j \in E$, the edge length L_{ij} satisfies

$$L_{ij} = \sqrt{R_i^2 + R_j^2 + 2\eta_{ij} R_i R_j}. \tag{3}$$

The map $R : V \to (0, +\infty)$ is referred to as a *Euclidean inversive distance circle packing* on $(S, \mathcal{T}, \eta)$. Specifically, the weight η_{ij} in (3) denotes the Euclidean inversive distance between the two Euclidean circles centered at v_i and v_j with radii R_i and R_j respectively.

2.2. **Inversive distance.** From the edge length formula (3), the Euclidean inversive distance $\eta(C_1, C_2)$ between two Euclidean circles C_1 and C_2 (with radii R_1 and R_2 respectively) is given by

$$\eta(C_1, C_2) = \frac{L_{12}^2 - R_1^2 - R_2^2}{2R_1 R_2}, \tag{4}$$

where L_{12} is the Euclidean distance between the centers of C_1 and C_2. If C_1 and C_2 intersect at an angle ϕ, then $\eta(C_1, C_2) = \cos\phi$. For disjoint C_1 and C_2, $\eta(C_1, C_2)$ equals the hyperbolic distance between hyperbolic planes in the upper half-space model of three-dimensional hyperbolic space $\mathbb{H}^3$, where these hyperbolic planes are realized as upper hemispheres passing through C_1 and C_2 respectively. For more details, we refer the reader to [4].

Analogously, from the edge length formula (1), the hyperbolic inversive distance $\eta(c_1, c_2)$ between two hyperbolic circles c_1 and c_2 (with radii r_1 and r_2 respectively) is given by

$$\eta(c_1, c_2) = \frac{\cosh l_{12} - \cosh r_1 \cosh r_2}{\sinh r_1 \sinh r_2}, \tag{5}$$

where l_{12} is the hyperbolic distance between the centers of c_1 and c_2. If c_1 and c_2 intersect at an angle ϕ, then $\eta(c_1, c_2) = \cos\phi$.

The Euclidean inversive distance in (4) and the hyperbolic inversive distance in (5) are related via stereographic projection. Specifically, we regard the unit sphere as a model of the hyperbolic plane $\mathbb{H}^2$ and the complex plane $\mathbb{C}$ as the ideal boundary of three-dimensional hyperbolic space $\mathbb{H}^3$. Taking $(0, 0, -1)$ as the projection center, stereographic projection maps hyperbolic circles c_1 and c_2 on $\mathbb{H}^2$ to Euclidean circles C_1 and C_2 on $\mathbb{C}$, respectively. Notably, both Euclidean and hyperbolic inversive distances can be expressed via the cross ratio (as detailed in [4]). Since stereographic projection preserves the cross ratio, the hyperbolic inversive distance between c_1 and c_2 equals the Euclidean inversive distance between their images C_1 and C_2, i.e., $\eta(c_1, c_2) = \eta(C_1, C_2)$. Furthermore, the isometry from the upper half-space model of $\mathbb{H}^3$ to the Poincaré disk $\mathbb{D}$ preserves this correspondence. Hence, for any two circles C_1 and C_2 in $\mathbb{D}$, their Euclidean inversive distance coincides with their hyperbolic inversive distance. We may thus directly identify hyperbolic inversive distance with Euclidean inversive distance.

For any Euclidean inversive distance circle packing R in the weighted triangulation $(\mathbb{D}, \mathcal{T}, \eta)$ of the Poincaré disk, the radius function R at each vertex corresponds to the radius of a Euclidean circle centered at that vertex. These Euclidean circles can also be interpreted as a hyperbolic circle packing in $\mathbb{D}$. Here, the vertices of the circle packing serve as the hyperbolic centers of the Euclidean circles. Two hyperbolic circles are adjacent if and only if their corresponding Euclidean circles are adjacent, which ensures their combinatorial structure matches $\mathcal{T}$. The hyperbolic radius function r is defined accordingly. Notably, the inversive distance is invariant under this correspondence. Furthermore, if the edge lengths of each triangle satisfy the strict triangle inequality (ensuring non-degenerate triangles), the Euclidean inversive distance circle packing naturally induces a hyperbolic inversive distance circle packing.

2.3. **Weighted Delaunay condition.** Let $\triangle v_1v_2v_3$ be a non-degenerate Euclidean triangle in $\mathbb{C}$ that is isometric to a face of the weighted triangulation $(S, \mathcal{T}, \eta)$. Each vertex v_i is associated with a circle of radius R_i centered at v_i, referred to as a vertex-circle. There exists a unique geometric center c_{123} with equal power distances to vertices v_1, v_2, v_3 (see [6, Proposition 7]). Here, the power distance from a point p to vertex v_i is defined as $\pi_i(p) = |p - v_i|^2 - R_i^2$, where $|p - v_i|$ denotes the Euclidean distance between p and v_i. The circle centered at c_{123} with radius $\sqrt{\pi_i(c_{123})}$ is called the face-circle of $\triangle v_1v_2v_3$, denoted C_{123}. Note that $\pi_i(c_{123})$ may be non-positive, in which case the face-circle is virtual, a situation that arises when $\eta_{12}, \eta_{13}, \eta_{23} \in (-1, 1]$. If the face-circle is real (i.e., not virtual), it is easy to verify that it is orthogonal to each vertex-circle.

Let $h_{ij,k}$ denote the signed distance from the geometric center c_{123} to the edge v_iv_j. This distance is positive if c_{123} lies on the same side of the line v_iv_j as $\triangle v_1v_2v_3$, negative if on the opposite side, and zero if c_{123} lies exactly on the line v_iv_j. For two adjacent non-degenerate Euclidean triangles $\triangle v_1v_2v_3$ and $\triangle v_1v_2v_4$ sharing the common edge v_1v_2, this edge is called *weighted Delaunay* in the PE metric if

$$h_{12,3} + h_{12,4} \geq 0. \tag{6}$$

If the face-circle C_{123} is a virtual circle, then (6) holds automatically. This conclusion follows directly from combining the specific expression of $h_{ij,k}$ (given in [8, 19, 20]) with (2). In contrast, when the face-circle C_{123} is non-virtual, the weighted Delaunay condition (6) admits a geometric interpretation. For the edge v_1v_2 in the PE metric, it is weighted Delaunay if and only if the vertex-circle centered at v_4 either does not intersect C_{123}, or intersects it at an exterior angle of at most $\frac{\pi}{2}$.

A triangulation $\mathcal{T}$ is called weighted Delaunay with respect to the PE metric if every interior edge is a weighted Delaunay edge. For simplicity, we refer to the Euclidean inversive distance circle packing R satisfying this condition as *weighted Delaunay*.

The definition of the weighted Delaunay condition in the hyperbolic case parallels its Euclidean counterpart. For a non-degenerate hyperbolic triangle $\triangle v_1v_2v_3$ induced by the radius function r via (1), there exists a geometric center c_{123} analogous to its Euclidean counterpart (see Glickenstein-Thomas [7] for details). Notably, this geometric center c_{123} may lie outside the hyperbolic plane. By projecting c_{123} onto the edges of the triangle $\triangle v_1v_2v_3$, the signed distance $h_{ij,k}$ from c_{123} to the edge v_iv_j can be defined in a similar manner. Explicit expressions for $h_{ij,k}$ in the hyperbolic setting are provided in [7, 19, 20]. For an edge v_1v_2 shared by two non-degenerate hyperbolic triangles $\triangle v_1v_2v_3$ and $\triangle v_1v_2v_4$ (both induced by r via (1)), the edge is called weighted Delaunay in the PH metric if

$$h_{12,3} + h_{12,4} \geq 0.$$

As in the Euclidean case, if the face-circle C_{123} is virtual, this condition is automatically satisfied. For non-virtual face-circles, the weighted Delaunay condition admits an analogous face-circle characterization: the edge v_1v_2 in the PH metric is weighted Delaunay if and only if the vertex-circle centered at v_4 either does not intersect C_{123}, or intersects it at an exterior angle of at most $\frac{\pi}{2}$.

Note that in the Poincaré disk $\mathbb{D}$, the intersection angles of hyperbolic circles coincide with those in the Euclidean background geometry. Additionally, a hyperbolic circle coincides with a Euclidean circle as a set, though their centers may not coincide. Hence, an edge is weighted Delaunay with respect to the PH metric if and only if it is weighted Delaunay with respect to the induced PE metric.

2.4. **Euclidean maximum principle.** A Euclidean triangle $\triangle v_1v_2v_3$ with edge lengths L_{12}, L_{13}, L_{23} is called a generalized triangle if its edge lengths satisfy the triangle inequality:

$$L_{12} \leq L_{23} + L_{13}, \quad L_{23} \leq L_{12} + L_{13}, \quad L_{13} \leq L_{12} + L_{23},$$

and the corresponding radius function R is referred to as a generalized Euclidean inversive distance circle packing. If all triangles in the weighted triangulation $(S, \mathcal{T}, \eta)$ are generalized triangles, we call R a generalized Euclidean inversive distance circle packing on $(S, \mathcal{T}, \eta)$. In particular, a generalized triangle $\triangle v_1v_2v_3$ is called degenerate if $L_{ij} = L_{ik} + L_{kj}$ for some permutation (i, j, k) of $\{1, 2, 3\}$.

Note that (6) applies only to non-degenerate Euclidean triangles. Luo-Xu-Zhang [14] extended the definition of the weighted Delaunay condition to generalized Euclidean triangles, and this extended version is used to prove the Euclidean maximum principle below.

Theorem 2.1 ([14], Theorem 3.1)**.** Let $\mathcal{T}$ be a star triangulation of P_n with boundary vertices $v_1, \ldots, v_n$ and a unique interior vertex v_0. Let η be a regular weight defined on the edges of $\mathcal{T}$ satisfying (2) with $\eta : E \to (-1, 1]$ or $\eta : E \to [0, +\infty)$. Suppose R and $\bar{R}$ are two generalized Euclidean inversive distance circle packings on $(P_n, \mathcal{T}, \eta)$ satisfying

(a): R and $\bar{R}$ are weighted Delaunay,

(b): the combinatorial curvatures $K_0(R)$ and $K_0(\bar{R})$ at the vertex v_0 satisfy $K_0(R) \leq K_0(\bar{R})$,

(c): $\max\left\{\frac{R_i}{\bar{R}_i} \mid i = 1, 2, \ldots, n\right\} \leq \frac{R_0}{\bar{R}_0}$.

Then there exists a constant $c > 0$ such that $R = c\bar{R}$.

Theorem 2.1 is a general theorem, as it holds for all generalized triangles. In fact, we only require it to hold for non-degenerate triangles. This is because when the PE metric and PH metric are mutually converted, the triangles involved are non-degenerate.

2.5. **Hyperbolic maximum principle and discrete Schwarz-Ahlfors lemma.** Let C_0 and C_1 be two circles contained in the Poincaré disk $\mathbb{D}$. Then C_0 and C_1 can be regarded both as Euclidean circles and hyperbolic circles, but the corresponding centers and radii in these two cases are usually different. Let R_0 and R_1 be their Euclidean radii, and let r_0 and r_1 be their hyperbolic radii. For the convenience of subsequent proofs, we may assume, without loss of generality, via a suitable Möbius transformation that C_0 is centered at the origin and C_1 is centered on the positive real axis. For any $\lambda \in (0, 1)$, let λC_0 and λC_1 be the images of C_0 and C_1 under the similarity transformation $f(z) = \lambda z$ on the complex plane $\mathbb{C}$, respectively. Clearly, λC_0 and λC_1 are still contained in $\mathbb{D}$. Let r_0^λ and r_1^λ be the hyperbolic radii of λC_0 and λC_1, respectively.

Lemma 2.2. Let $C_0, C_1, \lambda C_0$, and λC_1 be the circles defined above. The following statements hold:

(i): r_1 is a strictly increasing function of R_1.

(ii): For any $0 < \lambda < 1$, we have

$$\frac{\tanh(r_1^\lambda/2)}{\tanh(r_0^\lambda/2)} < \frac{\tanh(r_1/2)}{\tanh(r_0/2)}. \tag{7}$$

Proof. **(i)** Let l_{01} be the hyperbolic length of the edge from v_1 to the origin. By (1), we have

$$\cosh l_{01} = \cosh r_0 \cosh r_1 + \eta_{01} \sinh r_0 \sinh r_1.$$

It follows that

$$\begin{aligned} R_1 &= \frac{1}{2}(\tanh \frac{l_{01}+r_1}{2} - \tanh \frac{l_{01}-r_1}{2}) \\ &= \frac{\sinh r_1}{\cosh l_{01} + \cosh r_1} \\ &= \frac{1}{(\cosh r_0 + 1)\coth r_1 + \eta_{01}\sinh r_0}. \end{aligned}$$

This implies r_1 is a strictly increasing function of R_1.

(ii) Let z denote the intersection point of C_0 with the positive real axis. Then $0 < z < 1$. Let x, y denote the intersection points of C_1 with the real axis, where $|x| < y < 1$. Note that x may be negative. See Figure 3.

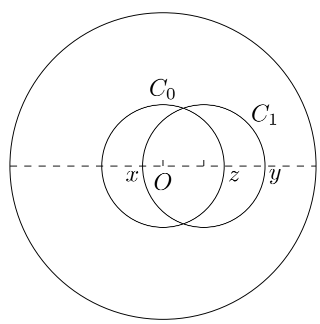

FIGURE 3. The circles C_0 and C_1 with $\eta_{01} \in (-1, 0)$.

Consider the function

$$f(\lambda) = \frac{\tanh(r_1^\lambda/2)}{\tanh(r_1/2)} - \frac{\tanh(r_0^\lambda/2)}{\tanh(r_0/2)}.$$

Note that $f(0) = f(1) = 0$. To obtain (7), it suffices to show $f(\lambda) < 0$ for $0 < \lambda < 1$. We will explicitly compute $f(\lambda)$ as follows.

In the Poincaré disk, the hyperbolic distance $d(a, b)$ of any two point a and b is defined by

$$\sinh \frac{d(a,b)}{2} = \frac{|a-b|}{\sqrt{(1-|a|^2)(1-|b|^2)}}. \tag{8}$$

For the circles C_0 and λC_0, we have

$$\sinh \frac{r_0}{2} = \frac{z}{\sqrt{1-z^2}} \quad \text{and} \quad \sinh \frac{r_0^\lambda}{2} = \frac{\lambda z}{\sqrt{1-(\lambda z)^2}}.$$

This implies that

$$\tanh\frac{r_0}{2} = z \quad \text{and} \quad \tanh\frac{r_0^\lambda}{2} = \lambda z. \tag{9}$$

Thus

$$\frac{\tanh(r_0^\lambda/2)}{\tanh(r_0/2)} = \lambda. \tag{10}$$

For the circles C_1 and λC_1, we have

$$\sinh r_1 = \frac{y-x}{\sqrt{1-x^2}\sqrt{1-y^2}} \quad \text{and} \quad \sinh r_1^\lambda = \frac{\lambda(y-x)}{\sqrt{1-(\lambda x)^2}\sqrt{1-(\lambda y)^2}}.$$

This implies that

$$\begin{aligned}\tanh\frac{r_1}{2} = \frac{\sinh r_1}{1+\cosh r_1} &= \frac{y-x}{\sqrt{(y-x)^2+(1-x^2)(1-y^2)}+\sqrt{(1-x^2)(1-y^2)}} \\ &= \frac{y-x}{1-xy+\sqrt{(1-x^2)(1-y^2)}}.\end{aligned} \tag{11}$$

Similarly,

$$\tanh\frac{r_1^\lambda}{2} = \frac{\lambda(y-x)}{1-\lambda^2 xy+\sqrt{(1-(\lambda x)^2)(1-(\lambda y)^2)}}.$$

Thus

$$\frac{\tanh(r_1^\lambda/2)}{\tanh(r_1/2)} = \frac{\lambda(1-xy+\sqrt{(1-x^2)(1-y^2)})}{1-\lambda^2 xy+\sqrt{(1-(\lambda x)^2)(1-(\lambda y)^2)}}. \tag{12}$$

Combining (10) and (12), we obtain

$$f(\lambda) = \frac{\lambda(1-xy+\sqrt{(1-x^2)(1-y^2)})}{1-\lambda^2 xy+\sqrt{(1-(\lambda x)^2)(1-(\lambda y)^2)}} - \lambda.$$

Define $Q_1 = (\lambda^2-1)xy$ and $Q_2 = \sqrt{(1-(\lambda x)^2)(1-(\lambda y)^2)} - \sqrt{(1-x^2)(1-y^2)}$. Rearranging the expression for $f(\lambda)$ gives

$$f(\lambda) = \frac{\lambda}{1-\lambda^2 xy+\sqrt{(1-(\lambda x)^2)(1-(\lambda y)^2)}}(Q_1-Q_2).$$

To complete the argument, it suffices to show $Q_1 < Q_2$ for $0 < \lambda < 1$. First, we rationalize Q_2 by multiplying numerator and denominator by the conjugate of the numerator:

$$\begin{aligned}Q_2 &= \frac{(1-\lambda^2x^2)(1-\lambda^2y^2)-(1-x^2)(1-y^2)}{\sqrt{(1-\lambda^2x^2)(1-\lambda^2y^2)}+\sqrt{(1-x^2)(1-y^2)}} \\ &= \frac{(1-\lambda^2)[x^2+y^2-(1+\lambda^2)x^2y^2]}{\sqrt{(1-\lambda^2x^2)(1-\lambda^2y^2)}+\sqrt{(1-x^2)(1-y^2)}}.\end{aligned} \tag{13}$$

Since $|x| < y < 1$, we consider two cases depending on the sign of xy:

Case 1: $xy \geq 0$. Note that $x^2+y^2 > 2xy \geq 2x^2y^2 \geq (1+\lambda^2)x^2y^2$. The denominator of Q_2 is clearly positive, so $Q_2 > 0$. Clearly, $Q_1 \leq 0$. Hence $Q_1 < Q_2$.

Case 2: $xy < 0$. Substituting $x^2 + y^2 > -2xy$ into the numerator of (13) yields

$$Q_2 > \frac{-(1-\lambda^2)xy[2+(1+\lambda^2)xy]}{\sqrt{(1-\lambda^2x^2)(1-\lambda^2y^2)} + \sqrt{(1-x^2)(1-y^2)}}.$$

Next, we bound the denominator from above by substituting $x^2 + y^2 > -2xy$ into the square roots:

$$\begin{aligned}&\sqrt{(1-\lambda^2x^2)(1-\lambda^2y^2)} + \sqrt{(1-x^2)(1-y^2)}\\ =&\sqrt{1-\lambda^2(x^2+y^2)+\lambda^4x^2y^2} + \sqrt{1-(x^2+y^2)+x^2y^2}\\ <&\,1+\lambda^2xy+1+xy\\ =&\,2+(1+\lambda^2)xy.\end{aligned}$$

Then

$$Q_2 > -(1-\lambda^2)xy = Q_1$$

Combining both cases, we conclude $Q_1 < Q_2$ for all $0 < \lambda < 1$. This completes the proof.

Q.E.D.

Using Lemma 2.2, we can prove the following theorem.

Theorem 2.3. Let η be a regular weight on $(P_n, \mathcal{T})$ satisfying the structure condition (2) with $\eta : E \to (-1, 1]$ or $\eta : E \to [0, +\infty)$. Suppose r and $\bar{r}$ are two weighted Delaunay hyperbolic inversive distance circle packings on $(P_n, \mathcal{T}, \eta)$ satisfying

(i) the combinatorial curvatures $K_0(r)$ and $K_0(\bar{r})$ at the vertex v_0 satisfy $K_0(r) \geq K_0(\bar{r})$,
(ii) all circles corresponding to r and $\bar{r}$ are contained in $\mathbb{D}$.

Define $u = \ln\tanh\frac{r}{2}$ and $\bar{u} = \ln\tanh\frac{\bar{r}}{2}$, and let $w_i = \bar{u}_i - u_i$ for each vertex $v_i \in V(\mathcal{T})$. Then the maximum value of w, i.e., $\max_{i\in\{0,1,\cdots,n\}} w_i = \max_{i\in\{0,1,\cdots,n\}}(\bar{u}_i - u_i)$, if > 0, is never achieved at v_0.

Proof. We prove this theorem by contradiction. Assume that

$$w_0 = \bar{u}_0 - u_0 = \max_{v_i\sim v_0}(\bar{u}_i - u_i) > 0.$$

By Möbius transformations, we may assume that v_0 is the origin. Set

$$\lambda = \frac{e^{u_0}}{e^{\bar{u}_0}} < 1.$$

By applying the similarity transformation $z \to \lambda z$ on the plane, we obtain a hyperbolic label $\bar{u}^\lambda$ induced by $\bar{u}$. By Lemma 2.2 (ii), for any $v_i \sim v_0$, we have $\bar{u}_i^\lambda - \bar{u}_0^\lambda < \bar{u}_i - \bar{u}_0$. From our assumption that $\bar{u}_0 - u_0 \geq \bar{u}_i - u_i$, it follows that

$$u_0 - u_i \leq \bar{u}_0 - \bar{u}_i < \bar{u}_0^\lambda - \bar{u}_i^\lambda. \tag{14}$$

A hyperbolic circle in $\mathbb{D}$ is also a Euclidean circle. Let R_v denote the Euclidean radius of the circle corresponding to vertex v. By (9), we have

$$e^{\bar{u}_0^\lambda} = \tanh\frac{\bar{r}_0^\lambda}{2} = \bar{R}_0^\lambda = \lambda\bar{R}_0 = \lambda\tanh\frac{\bar{r}_0}{2} = \lambda e^{\bar{u}_0}. \tag{15}$$

By our choice of λ, this implies $e^{\bar{u}_0^\lambda} = e^{u_0}$, so $\bar{u}_0^\lambda = u_0$. From (14), we further get $u_i > \bar{u}_i^\lambda$. Hence, $\bar{r}_0^\lambda = r_0$ and $r_i > \bar{r}_i^\lambda$. Then $R_0 = \bar{R}_0^\lambda$, and $R_i > \bar{R}_i^\lambda$ by Lemma 2.2 (i). Note that R_v and R_v^λ both satisfy the weighted Delaunay condition.

In a 1-ring neighborhood of v_0, it holds that $R_0 = \bar{R}_0^\lambda$ and $R_i > \bar{R}_i^\lambda$ for all $v_i \sim v_0$. This implies that the maximum of $\bar{R}_i^\lambda/R_i$ is attained at the interior vertex v_0. Note that the hyperbolic angle at the origin of a hyperbolic triangle coincides with the Euclidean angle at the origin of its corresponding Euclidean triangle. Consequently, the hyperbolic combinatorial curvature at v_0 equals the Euclidean combinatorial curvature at v_0. By our assumption, $K_0(R) \geq K_0(\bar{R})$. Since Euclidean angles are invariant under similarity transformations, it follows that $K_0(\bar{R}) = K_0(\bar{R}^\lambda)$. Therefore, $K_0(R) \geq K_0(\bar{R}^\lambda)$. By Theorem 2.1, $\bar{R}_i^\lambda / R_i = \bar{R}_0^\lambda / R_0$ for all $v_i \sim v_0$. Hence, $R_i = \bar{R}_i^\lambda$. This contradicts $R_i > \bar{R}_i^\lambda$, completing the proof. Q.E.D.

Proof of Theorem 1.2: Part (i) follows directly from Theorem 2.3. We derive Part (ii) by substituting w_0 with $-w_0$ in Part (i). This substitution is valid due to the assumption $K_0(r) \leq K_0(\bar{r})$. Q.E.D.

As an application of Theorem 2.3, we obtain the following discrete Schwarz-Ahlfors lemma.

Theorem 2.4 (Discrete Schwarz-Ahlfors lemma)**.** Let η be a regular weight on $(M, \mathcal{T})$ satisfying the structure condition (2) with $\eta : E \to (-1, 1]$ or $\eta : E \to [0, +\infty)$, where $M \subseteq \mathbb{D}$ is a compact set with non-empty boundary. Let r and $\bar{r}$ be two weighted Delaunay hyperbolic inversive distance circle packings on $(M, \mathcal{T}, \eta)$, with all corresponding circles contained in $\mathbb{D}$. Then the following statements hold:

(i): If the combinatorial curvatures $K(r) \geq K(\bar{r})$ for all interior vertices, and $w \leq 0$ holds for every boundary vertex, then $w \leq 0$ holds for all vertices.

(ii): If the combinatorial curvatures $K(r) \leq K(\bar{r})$ for all interior vertices, and $w \geq 0$ holds for every boundary vertex, then $w \geq 0$ holds for all vertices.

Proof. Part (i) follows directly from Theorem 1.2 (i). We prove it by contradiction. Suppose there exists an interior vertex v_i with $w_i > 0$. Then w attains its maximum at some interior vertex. We may assume without loss of generality that $w_i = \max_j w_j > 0$. By applying Theorem 1.2 (i) to the 1-ring neighborhood of v_i, we deduce that there exists a vertex $v_j \sim v_i$ with $w_j > w_i$. This contradicts the maximality of w_i. Part (ii) follows analogously from Theorem 1.2 (ii) by a similar argument. Q.E.D.

Theorem 1.4 is a direct corollary of Theorem 2.4. Also, we have the following result related to the rigidity of hyperbolic inversive distance circle packings.

Corollary 2.5. Under the same conditions as in Theorem 2.4, if $K(r) \equiv K(\bar{r})$ for all interior vertices, and $w \equiv 0$ holds for every boundary vertex, then $w \equiv 0$ holds for all vertices.

Remark 2.6. An alternative approach to proving Corollary 2.5 involves constructing convex energy functions, as described in [19, 20]. Furthermore, when using this method to establish rigidity, there is no need to assume that the two PH metrics r and $\bar{r}$ are weighted Delaunay. For further details, we refer the reader to [19, 20].

3. Infinite rigidity of hyperbolic inversive distance circle packings

In the previous section, we assume that all circles are contained in the Poincaré disk $\mathbb{D}$. In this section, we remove this constraint, where the circles may intersect $\partial\mathbb{D}$ or even lie outside it, and further generalize Theorem 2.3 and Theorem 2.4.

In Subsection 2.2, we have shown that the Euclidean inversive distance and hyperbolic inversive distance between any two circles contained in $\mathbb{D}$ are equal. As a natural generalization, we extend the notion of hyperbolic circles to include Euclidean circles that intersect $\partial\mathbb{D}$ or lie outside $\mathbb{D}$. For simplicity, we refer to these as "generalized hyperbolic circles". The hyperbolic inversive distance for generalized hyperbolic circles is defined as their corresponding Euclidean inversive distance.

Definition 3.1. Given a vertex v and its corresponding circle C_v, we define its generalized hyperbolic radius ρ_v as follows:

(i) If C_v is contained in $\mathbb{D}$, then $\rho_v = \tanh \frac{r_v}{2}$, where r_v is the hyperbolic radii of C_v;

(ii) If C_v intersects $\partial\mathbb{D}$ or lies outside $\mathbb{D}$, let η_v denote the Euclidean inversive distance between C_v and $\partial\mathbb{D}$ (the unit circle). Then

$$\eta_v = \frac{L_v^2 - R_v^2 - 1}{2R_v}, \tag{16}$$

where R_v is the Euclidean radius of C_v, and L_v is the Euclidean distance between the Euclidean center of C_v and the origin. In this case, ρ_v is defined as ∞^{η_v+1}, where the exponent is non-negative as $\eta_v \in [-1, +\infty)$. Specifically, $\eta_v = -1$ corresponds to C_v being internally tangent to $\partial\mathbb{D}$.

We adopt the convention that for any $\beta \geq \alpha \geq 0$ and any positive real number a, $\infty^\beta \geq \infty^\alpha > a$ and $a/\infty^\alpha = 0$.

Let C_0 be a circle centered at the origin and contained in $\mathbb{D}$, and let C_1 be a generalized hyperbolic circle adjacent to C_0 with their inversive distance $\eta_{01} \in (-1, +\infty)$. Let R_0 and R_1 be their Euclidean radii, and let ρ_0 and ρ_1 be their generalized hyperbolic radii. For any $\lambda \in (0, 1)$, let λC_0 and λC_1 be the images of C_0 and C_1 under the similarity transformation $f(z) = \lambda z$ on the complex plane $\mathbb{C}$, respectively. Clearly, λC_0 remains contained in $\mathbb{D}$. Assume that λC_1 is also contained in $\mathbb{D}$. Let ρ_0^λ and ρ_1^λ be the generalized hyperbolic radii of λC_0 and λC_1, respectively.

Lemma 3.2. Let $C_0, C_1, \lambda C_0$, and λC_1 be the circles defined above. The following statements hold:

(i): ρ_1 is a strictly increasing function of R_1.
(ii): For any $0 < \lambda < 1$, we have

$$\frac{\rho_1^\lambda}{\rho_0^\lambda} < \frac{\rho_1}{\rho_0}. \tag{17}$$

Proof. **(i)** If C_1 is contained in $\mathbb{D}$, then the conclusion follows directly from Lemma 2.2 (i). When C_1 gradually expands from the interior of $\mathbb{D}$ to being internally tangent to $\partial\mathbb{D}$, it is straightforward to see that ρ_1 increases in this case. Now suppose C_1 is not contained in $\mathbb{D}$. Let L be the Euclidean distance between the Euclidean center of C_1 and the origin. By (3), it follows that $L^2 = R_0^2 + R_1^2 + 2\eta_{01}R_0R_1$. By (16), we have

$$\eta_1 = \frac{R_0^2 + 2R_0R_1\eta_{01} - 1}{2R_1} = \frac{R_0^2 - 1}{2R_1} + R_0\eta_{01}.$$

This implies η_1 is a strictly increasing function of R_1. Hence, $\rho_1 = \infty^{\eta_1+1}$ is strictly increasing in R_1.

(ii) If C_1 is contained in $\mathbb{D}$, then the conclusion follows directly from Lemma 2.2 (ii). Otherwise, the conclusion follows from the convention specified in Definition 3.1. Q.E.D.

The following theorem generalizes Theorem 2.3.

Theorem 3.3. Let η be a regular weight on $(P_n, \mathcal{T})$ satisfying the structure condition (2) with $\eta : E \to (-1, 1]$ or $\eta : E \to [0, +\infty)$. Suppose r and $\bar{r}$ are two weighted Delaunay hyperbolic inversive distance circle packings on $(P_n, \mathcal{T}, \eta)$ satisfying

(i) the combinatorial curvatures $K_0(r)$ and $K_0(\bar{r})$ at the vertex v_0 satisfy $K_0(r) \geq K_0(\bar{r})$,

(ii) all circles corresponding to r have non-empty intersection with $\mathbb{D}$, and all circles corresponding to $\bar{r}$ are contained in $\mathbb{D}$.

Set

$$e^{w_v} = \frac{\bar{\rho}_v}{\rho_v},$$

where ρ_v and $\bar{\rho}_v$ denote the generalized hyperbolic radii corresponding to r_v and $\bar{r}_v$, respectively. Then the maximum of w, if > 0, is never achieved at v_0.

Proof. We prove this theorem by contradiction. Assume there exists an interior vertex v_0 such that

$$e^{w_0} = \frac{\bar{\rho}_0}{\rho_0} = \max_{v_i \sim v_0} \frac{\bar{\rho}_i}{\rho_i} > 1.$$

Then ρ_0 must be a real number. Otherwise, $\bar{\rho}_0 > \rho_0$ would imply $\bar{\rho}_0$ is not real, which contradicts the assumption that all circles corresponding to $\bar{r}$ are contained in $\mathbb{D}$.

By Möbius transformations, we map v_0 to the origin. Since both ρ_0 and $\bar{\rho}_0$ are real, their corresponding circles are contained in $\mathbb{D}$. Set

$$\lambda = \frac{\rho_0}{\bar{\rho}_0} < 1.$$

Applying the similarity transformation $z \to \lambda z$ on the plane, we obtain the generalized hyperbolic radii $\bar{\rho}^\lambda$ induced by $\bar{\rho}$. By Lemma 3.2 (ii), for any $v_i \sim v_0$, we have

$$\frac{\bar{\rho}_i^\lambda}{\bar{\rho}_0^\lambda} < \frac{\bar{\rho}_i}{\bar{\rho}_0} \leq \frac{\rho_i}{\rho_0}.$$

By (15), it follows that $\bar{\rho}_0^\lambda = \lambda \bar{\rho}_0 = \rho_0$. This further implies $\bar{\rho}_i^\lambda < \rho_i$.

Let R_v denote the Euclidean radius of the circle corresponding to vertex v. Then $\bar{\rho}_0^\lambda = \rho_0$ implies $\bar{R}_0^\lambda = R_0$, and by Lemma 3.2 (i), $\bar{\rho}_i^\lambda < \rho_i$ implies $\bar{R}_i^\lambda < R_i$. The rest of the proof is the same as that of Theorem 2.3, so we omit it here. Q.E.D.

As a direct corollary of Theorem 3.3, we obtain the following discrete Schwarz-Ahlfors lemma.

Theorem 3.4 (Discrete Schwarz-Ahlfors lemma)**.** Let η be a regular weight on $(M, \mathcal{T})$ satisfying the structure condition (2) with $\eta : E \to (-1, 1]$ or $\eta : E \to [0, +\infty)$, where $M \subseteq \mathbb{D}$ is a compact set with non-empty boundary. Let r and $\bar{r}$ be two weighted Delaunay hyperbolic inversive distance circle packings on $(M, \mathcal{T}, \eta)$, where all circles corresponding to $\bar{r}$ are contained in $\mathbb{D}$. Denote $w_v = \ln \frac{\bar{\rho}_v}{\rho_v}$, where ρ_v and $\bar{\rho}_v$ are the generalized hyperbolic radii corresponding to r_v and $\bar{r}_v$, respectively. If the combinatorial curvatures $K(r) \geq K(\bar{r})$ for all interior vertices, and $w \leq 0$ holds for every boundary vertex, then $w \leq 0$ holds for all vertices.

Proof of Theorem 1.5: We prove this theorem by contradiction. Suppose there exists a vertex $v_0 \in V(\mathcal{T})$ such that $r_0 < \bar{r}_0$, which is equivalent to $\rho_0 < \bar{\rho}_0$. Here ρ_v and $\bar{\rho}_v$ are the generalized hyperbolic radii corresponding to r_v and $\bar{r}_v$, respectively.

There exists a constant $\mu = 1 + \epsilon > 1$ such that the inversive distance circle packing $(\mathcal{T}, \mu r)$, obtained by scaling $(\mathcal{T}, r)$ under the similarity transformation $z \mapsto \mu z$, satisfies

$$\rho_0^\mu < \bar{\rho}_0, \tag{18}$$

and remains weighted Delaunay.

Let $(\mathcal{T}_1, \mu r)$ be the sub-circle packing of $(\mathcal{T}, \mu r)$, consisting of all circles contained in $\mathbb{D}$ together with those in their 1-ring neighborhoods that are not contained in $\mathbb{D}$, i.e., either intersecting $\partial\mathbb{D}$ or lying outside $\mathbb{D}$. Here, $\mathcal{T}_1$ denotes the corresponding triangulation. Since $\mathcal{T}$ is locally finite, $\mathcal{T}_1$ is also locally finite. For each boundary vertex $v_B \in V(\mathcal{T}_1)$, the corresponding circle in $(\mathcal{T}_1, \mu r)$ intersects $\partial\mathbb{D}$ or lies outside $\mathbb{D}$, so $\rho^\mu_B > \bar{\rho}_B$.

Applying Theorem 3.4 to $(\mathcal{T}_1, \mu r)$ and $(\mathcal{T}, \bar{r})$, we have $\rho^\mu_v > \bar{\rho}_v$ for all $v \in V(\mathcal{T}_1)$. Since $v_0 \in V(\mathcal{T}_1)$ by the definition of $\mathcal{T}_1$, this implies $\rho^\mu_0 \geq \bar{\rho}_0$, which contradicts (18). This completes the proof. Q.E.D.

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, 74074, U.S.
Email address: yanwen.luo@okstate.edu

SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, WUHAN, 430072, P.R.CHINA
Email address: xuxu2@whu.edu.cn

INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH (BICMR), BEIJING (PEKING) UNIVERSITY, BEIJING, 100871, P.R. CHINA
Email address: chaozheng@pku.edu.cn