GENERALIZED CURVATURES OF CURVES IN \mathbb{R}^n

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ABSTRACT. For a curve $\gamma: I \to \mathbb{R}^n$ of order n-1, we prove that the generalized curvatures $\kappa_1, \ldots, \kappa_{n-1}$ can be expressed in terms of the leading principal minors of the matrix $\mathbf{A}(t)^T \mathbf{A}(t)$, where $\mathbf{A}(t)$ is the $n \times n$ matrix whose i-th column is $\gamma^{(i)}(t)$. This gives an efficient algorithm to calculate the curvatures.

1. Introduction

A parametrized curve in \mathbb{R}^n is a smooth function $\gamma: I \to \mathbb{R}^n$ defined on an open interval I, such that $\gamma'(t) \neq \mathbf{0}$ for all $t \in I$. The image of γ is a one-dimensional Riemannian manifold with metric induced by the Euclidean metric of \mathbb{R}^n .

If $\gamma: I \to \mathbb{R}^n$ is a parametrized curve such that the vectors $\gamma'(t), \ldots, \gamma^{(n-1)}(t)$ are linearly independent, we say that the curve has order n-1. For such a curve, one can define the Frenet frame $\{\mathbf{T}(t), \mathbf{N}_1(t), \ldots, \mathbf{N}_{n-1}(t)\}$ for each $t \in I$ using the Gram-Schmidt process and the generalized cross product. The matrix

$$\mathbf{F} = \left[\mathbf{T}(t) \mid \mathbf{N}_1(t) \mid \dots \mid \mathbf{N}_{n-1}(t) \right]$$

is an orthogonal matrix with determinant 1. The Frenet-Serret formulas say that with respect to the arclength parameter s,

$$\frac{d\mathbf{T}}{ds} = \kappa_1 \mathbf{N}_1,$$

$$\frac{d\mathbf{N}_1}{ds} = -\kappa_1 \mathbf{T} + \kappa_2 \mathbf{N}_2,$$

$$\vdots$$

$$\frac{d\mathbf{N}_{n-2}}{ds} = -\kappa_{n-2} \mathbf{N}_{n-3} + \kappa_{n-1} \mathbf{N}_{n-1},$$

$$\frac{d\mathbf{N}_{n-1}}{ds} = -\kappa_{n-1} \mathbf{N}_{n-2}.$$

The numbers $\kappa_1, \ldots, \kappa_{n-1}$ are the generalized curvatures of the curve.

In the classical case where n=3, $\kappa_1=\kappa$ and $\kappa_2=\tau$ are respectively the curvature and the torsion of the curve. The fundamental theorem of the local theory of curves asserts that κ and τ uniquely determine the curve up to a direct isometry of \mathbb{R}^3 .

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Given a space curve $\gamma:I\to\mathbb{R}^3$ in any parametrization, the curvature κ and the torsion τ can be computed by the formulas

$$\kappa = \frac{\|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\|}{\|\boldsymbol{\gamma}'\|^3}, \qquad \tau = \frac{\langle \boldsymbol{\gamma}' \times \boldsymbol{\gamma}'', \boldsymbol{\gamma}''' \rangle}{\|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\|^2}. \tag{1.1}$$

In this work, we establish an efficient algorithm to compute the generalized curvatures $\kappa_1, \ldots, \kappa_{n-1}$ of a parametrized curve $\gamma : I \to \mathbb{R}^n$ of order n-1 in terms of $\gamma'(t), \ldots, \gamma^{(n)}(t)$, generalizing the formulas (1.1) to higher dimensions. Let $\mathbf{A}(t)$ be the matrix

$$\mathbf{A}(t) = \begin{bmatrix} \boldsymbol{\gamma}'(t) & \boldsymbol{\gamma}''(t) & \cdots & \boldsymbol{\gamma}^{(n)}(t) \end{bmatrix}.$$

For $1 \le i \le n$, define $\mathbf{M}_i(t)$ as the $i \times i$ matrix

$$\mathbf{M}_i(t) = egin{bmatrix} \langle oldsymbol{\gamma}'(t), oldsymbol{\gamma}'(t)
angle & \cdots & \langle oldsymbol{\gamma}'(t), oldsymbol{\gamma}^{(i)}(t)
angle \ dots & \ddots & dots \ \langle oldsymbol{\gamma}^{(i)}(t), oldsymbol{\gamma}'(t)
angle & \cdots & \langle oldsymbol{\gamma}^{(i)}(t), oldsymbol{\gamma}^{(i)}(t)
angle \end{bmatrix},$$

which consists of the first i rows and first i columns of the matrix $\mathbf{B}(t) = \mathbf{A}(t)^T \mathbf{A}(t)$. We show that for $1 \le i \le n-1$,

$$\kappa_i(t)^2 = \frac{\det \mathbf{M}_{i+1}(t) \det \mathbf{M}_{i-1}(t)}{\left[\|\boldsymbol{\gamma}'(t)\| \det \mathbf{M}_i(t)\right]^2}.$$
(1.2)

From this, we can obtain $\kappa_i(t)$ using the fact that for $1 \le i \le n-2$, $\kappa_i(t)$ is positive. For i = n-1, $\kappa_{n-1}(t)$ has the same sign as $\det \mathbf{A}(t)$, and $(\det \mathbf{A}(t))^2 = \det \mathbf{M}_n(t)$.

2. Preliminaries

First, we give a brief revision of the linear algebra on the Euclidean space \mathbb{R}^n , fixing the notations. A vector \mathbf{v} in \mathbb{R}^n is denoted as

$$\mathbf{v} = (v_1, \dots, v_n)$$
 or $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$.

In terms of the standard unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$, $\mathbf{v} = v_1 \mathbf{e}_1 + \dots + v_n \mathbf{e}_n$. If $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ are two vectors in \mathbb{R}^n , their Euclidean inner product is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n u_i v_i = \mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u}.$$

The norm of the vector \mathbf{v} is defined as

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

A basis $\{\mathbf v_1,\dots,\mathbf v_n\}$ of $\mathbb R^n$ is said to be positively oriented if

$$\det \left[\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_n \right] > 0.$$

Given a linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ in \mathbb{R}^n , the Gram-Schidmt process produces an orthonormal set $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ such that for $1 \leq j \leq r$,

$$span\{\mathbf{u}_1,\ldots,\mathbf{u}_j\}=span\{\mathbf{v}_1,\ldots,\mathbf{v}_j\}.$$

If $R_{i,j} = \langle \mathbf{u}_i, \mathbf{v}_j$, then $\mathbf{R} = [R_{i,j}]$ is an upper triangular matrix with positive diagonal entries, and

$$\begin{bmatrix} \mathbf{v}_1 \mid \cdots \mid \mathbf{v}_r \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \mid \cdots \mid \mathbf{u}_r \end{bmatrix} \mathbf{R}.$$

Classically, the cross product is defined for two vectors in \mathbb{R}^3 . It can be generalized in the following way.

Definition 2.1 (Generalized Cross Product). For $n \geq 2$, the generalized cross product of n-1 vectors in \mathbb{R}^n is a (n-1)-linear map $\mathscr{P}: (\mathbb{R}^n)^{n-1} \to \mathbb{R}^n$. Given n-1 vectors $\mathbf{v}_j = (v_{1,j}, v_{2,j}, \ldots, v_{n,j}), 1 \leq j \leq n-1$, their cross product $\mathscr{P}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$ is the vector

$$\mathscr{P}(\mathbf{v}_1,\ldots,\mathbf{v}_{n-1}) = \det \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{n-1} & \mathbf{e}_1 \\ \vdots & \vdots & \vdots \\ \mathbf{e}_n \end{bmatrix} = \det \begin{bmatrix} v_{1,1} & \cdots & v_{1,n-1} & \mathbf{e}_1 \\ v_{2,1} & \cdots & v_{2,n-1} & \mathbf{e}_2 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n,1} & \cdots & v_{n,n-1} & \mathbf{e}_n \end{bmatrix}.$$

In the determinant, $\mathbf{e}_1, \dots, \mathbf{e}_n$ are treated as formal symbols. The components of the cross product $\mathcal{P}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ along $\mathbf{e}_1, \dots, \mathbf{e}_n$ are obtained by computing this determinant using column expansion with respect to the last column.

By the properties of determinants, the (n-1)-linear map $\mathscr{P}: (\mathbb{R}^n)^{n-1} \to \mathbb{R}^n$ is an alternating linear mapping. Namely, if σ is a permutation of the set $\{1, 2, \dots, n-1\}$, then

$$\mathscr{P}\left(\mathbf{v}_{\sigma(1)},\mathbf{v}_{\sigma(2)},\ldots,\mathbf{v}_{\sigma(n-1)}\right)=\mathrm{sgn}(\sigma)\mathscr{P}(\mathbf{v}_1,\ldots,\mathbf{v}_{n-1}).$$

When n = 3, $\mathscr{P}(\mathbf{u}, \mathbf{v})$ is just the ordinary cross product of \mathbf{u} and \mathbf{v} .

The generalized cross product has certain properties whose proofs are straightforward generalizations of those for the n=3 case.

Proposition 2.2. Let $n \geq 2$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ be vectors in \mathbb{R}^n . Then the generalized cross product $\mathscr{P}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$ is zero if and only if the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ is linearly dependent.

Proposition 2.3. Let $n \geq 2$, and let $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_n$ be vectors in \mathbb{R}^n . Assume that $\mathbf{v}_j = (v_{1,j}, \dots, v_{n,j})$ for $1 \leq j \leq n$. Then

$$\langle \mathscr{P}(\mathbf{v}_1,\ldots,\mathbf{v}_{n-1}),\mathbf{v}_n\rangle = \det \left[\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_{n-1} \mid \mathbf{v}_n\right] = \det \left[v_{i,j}\right].$$

Proposition 2.3 gives the following important properties of cross product.

Corollary 2.4. Let $n \geq 2$. If $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ are vectors in \mathbb{R}^n , the cross product $\mathscr{P}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ is orthogonal to each of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$.

Corollary 2.5. Let $n \geq 2$, and let $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ be vectors in \mathbb{R}^n . Denote by $\mathbf{w} = \mathscr{P}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ their cross product. If $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ is a linearly independent set, then $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{w}\}$ is a positively oriented basis of \mathbb{R}^n .

For the norm of the vector $\mathscr{P}(\mathbf{v}_1,\ldots,\mathbf{v}_{n-1})$, we first prove the following general formula.

Proposition 2.6. Let $n \geq 2$. If $\{\mathbf{u}_1, \dots, \mathbf{u}_{n-1}\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ are two sets of vectors in \mathbb{R}^n , let

$$\mathbf{u} = \mathscr{P}(\mathbf{u}_1, \dots, \mathbf{u}_{n-1}), \qquad \mathbf{v} = \mathscr{P}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}).$$

Then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \det \left[\langle \mathbf{u}_i, \mathbf{v}_j \rangle \right] = \det \left[\begin{pmatrix} \langle \mathbf{u}_1, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_1, \mathbf{v}_{n-1} \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{u}_{n-1}, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_{n-1}, \mathbf{v}_{n-1} \rangle \end{pmatrix} \right].$$

Proof. If $\mathbf{v} = \mathbf{0}$, then $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. Proposition 2.2 says that $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ is a linearly dependent set. Therefore, there exists a nonzero vector $\mathbf{c} = (c_1, \dots, c_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1} = \mathbf{0}.$$

Then

$$\begin{bmatrix} \langle \mathbf{u}_1, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_1, \mathbf{v}_{n-1} \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{u}_{n-1}, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_{n-1}, \mathbf{v}_{n-1} \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{u}_1, c_1 \mathbf{v}_1 + \cdots + c_{n-1} \mathbf{v}_{n-1} \rangle \\ \vdots \\ \langle \mathbf{u}_{n-1}, c_1 \mathbf{v}_1 + \cdots + c_{n-1} \mathbf{v}_{n-1} \rangle \end{bmatrix} = \mathbf{0}.$$

This implies that the matrix $\left[\langle \mathbf{u}_i, \mathbf{v}_j \rangle\right]$ is singular. Hence,

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0 = \det \left[\langle \mathbf{u}_i, \mathbf{v}_j \rangle \right]$$

holds. If $\mathbf{v} \neq \mathbf{0}$, then $\langle \mathbf{v}, \mathbf{v} \rangle > 0$. By Proposition 2.3,

$$\langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{v}, \mathbf{v} \rangle = \det \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_{n-1} & \mathbf{v} \end{bmatrix}^T \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{n-1} & \mathbf{v} \end{bmatrix}$$

$$= \det \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_1, \mathbf{v}_{n-1} \rangle & \langle \mathbf{u}_1, \mathbf{v} \rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle \mathbf{u}_{n-1}, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_{n-1}, \mathbf{v}_{n-1} \rangle & \langle \mathbf{u}_{n-1}, \mathbf{v} \rangle \\ \langle \mathbf{v}, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{v}, \mathbf{v}_{n-1} \rangle & \langle \mathbf{v}, \mathbf{v} \rangle \end{bmatrix}.$$

By Corollary 2.4, v is orthogonal to v_1, \ldots, v_{n-1} . Therefore,

$$\langle \mathbf{v}, \mathbf{v}_1 \rangle = \cdots = \langle \mathbf{v}, \mathbf{v}_{n-1} \rangle = 0.$$

Thus,

$$\langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{v}, \mathbf{v} \rangle = \det \begin{bmatrix} \langle \mathbf{u}_{1}, \mathbf{v}_{1} \rangle & \cdots & \langle \mathbf{u}_{1}, \mathbf{v}_{n-1} \rangle & \langle \mathbf{u}_{1}, \mathbf{v} \rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle \mathbf{u}_{n-1}, \mathbf{v}_{1} \rangle & \cdots & \langle \mathbf{u}_{n-1}, \mathbf{v}_{n-1} \rangle & \langle \mathbf{u}_{n-1}, \mathbf{v} \rangle \\ 0 & \cdots & 0 & \langle \mathbf{v}, \mathbf{v} \rangle \end{bmatrix}$$
$$= \langle \mathbf{v}, \mathbf{v} \rangle \det \begin{bmatrix} \langle \mathbf{u}_{1}, \mathbf{v}_{1} \rangle & \cdots & \langle \mathbf{u}_{1}, \mathbf{v}_{n-1} \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{u}_{n-1}, \mathbf{v}_{1} \rangle & \cdots & \langle \mathbf{u}_{n-1}, \mathbf{v}_{n-1} \rangle \end{bmatrix}.$$

Since $\langle \mathbf{v}, \mathbf{v} \rangle > 0$, we find that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \det \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_1, \mathbf{v}_{n-1} \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{u}_{n-1}, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{u}_{n-1}, \mathbf{v}_{n-1} \rangle \end{bmatrix}.$$

Taking $\mathbf{u}_j = \mathbf{v}_j$ for $1 \leq j \leq n-1$ in Proposition 2.6, we obtain the norm of the vector $\mathscr{P}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$.

Corollary 2.7. Let $n \geq 2$, and let $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ be vectors in \mathbb{R}^n . The cross product $\mathscr{P}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ is a vector in \mathbb{R}^n with norm

$$\|\mathscr{P}(\mathbf{v}_1,\ldots,\mathbf{v}_{n-1})\| = \sqrt{\det\left[\langle \mathbf{v}_i,\mathbf{v}_j\rangle\right]}.$$

From this, we obtain the following.

Corollary 2.8. Let $n \geq 2$, and let $\{\mathbf{u}_1, \dots, \mathbf{u}_{n-1}\}$ be an orthonormal set in \mathbb{R}^n . If

$$\mathbf{u}_n = \mathscr{P}(\mathbf{u}_1, \dots, \mathbf{u}_{n-1}),$$

then $\{\mathbf{u}_1,\ldots,\mathbf{u}_{n-1},\mathbf{u}_n\}$ is a positively oriented orthonormal basis of \mathbb{R}^n .

Next, we discuss the Euclidean geometry of curves in \mathbb{R}^n . Some standard textbooks in this topic are [Ger62, Kli78, dC16, Tap16, GAS06, KÏ5, Spi79a, Spi79b].

A parametrized curve in \mathbb{R}^n is a smooth function $\gamma: I \to \mathbb{R}^n$ defined on an open interval I with $\gamma'(t) \neq 0$ for all $t \in I$. Fixed a $t_0 \in I$, then the arclength function $s: I \to \mathbb{R}$ defined by

$$s(t) = \int_{t_0}^t \|\boldsymbol{\gamma}'(\tau)\| d\tau$$

is a strictly increasing smooth function. If $\tilde{\gamma}: J \to \mathbb{R}^n$ is the function defined as

$$\widetilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \circ s^{-1},$$

then $\widetilde{\gamma}: J \to \mathbb{R}^n$ is a reparametrization of $\gamma: I \to \mathbb{R}^n$ by arc-length. We usually use s instead of t as the parameter for a curve $\widetilde{\gamma}: J \to \mathbb{R}^n$ that is parametrized by arclength.

Definition 2.9 (The Canonical Matrix). Let $\gamma: I \to \mathbb{R}^n$ be a parametrized curve. The canonical matrix of γ at $t \in I$ is defined to be the $n \times n$ matrix

$$\mathbf{A}(t) = \left[\boldsymbol{\gamma}'(t) \mid \boldsymbol{\gamma}''(t) \mid \cdots \mid \boldsymbol{\gamma}^{(n)}(t) \right].$$

If $\widetilde{\gamma}: J \to \mathbb{R}^n$ is a reparametrization of $\gamma: I \to \mathbb{R}^n$, there exists a strictly increasing diffeomorphism $\phi: J \to I$ such that

$$\widetilde{\gamma}(t) = \gamma(\phi(t)).$$

In particular, $\phi'(t) > 0$ for all $t \in J$. We have the following.

Proposition 2.10. Let $\gamma:I\to\mathbb{R}^n$ be a parametrized curve. Assume that $\widetilde{\gamma}:J\to\mathbb{R}^n$ is a reparametrization of $\gamma:I\to\mathbb{R}^n$ such that $\widetilde{\gamma}(t)=\gamma(\phi(t))$ for a strictly increasing diffeomorphism $\phi:J\to I$. Then for $1\leq j\leq n$, there exist smooth functions $U_{i,j}(t)$, $1\leq i\leq j$ such that

$$\widetilde{\boldsymbol{\gamma}}^{(j)}(t) = \sum_{i=1}^{j} U_{i,j}(t) \boldsymbol{\gamma}^{(i)}(\phi(t)),$$

with $U_{j,j}(t) = \phi'(t)^j$. In other words, if

$$\mathbf{A}(t) = \left[\boldsymbol{\gamma}'(t) \mid \boldsymbol{\gamma}''(t) \mid \cdots \mid \boldsymbol{\gamma}^{(n)}(t) \right] \quad \text{and} \quad \widetilde{\mathbf{A}}(t) = \left[\widetilde{\boldsymbol{\gamma}}'(t) \mid \widetilde{\boldsymbol{\gamma}}''(t) \mid \cdots \mid \widetilde{\boldsymbol{\gamma}}^{(n)}(t) \right]$$

are the canonical matrices of $\gamma(t)$ and $\widetilde{\gamma}(t)$, there is an upper triangular matrix $\mathbf{U}(t) = [U_{i,j}(t)]$ with diagonal entries $U_{j,j}(t) = \phi'(t)^j$, $1 \le j \le n$, such that

$$\widetilde{\mathbf{A}}(t) = \mathbf{A}(\phi(t))\mathbf{U}(t).$$

The regularity order of a curve is defined in the following way.

Definition 2.11 (The Regularity Order of a Curve). Given a curve $\gamma: I \to \mathbb{R}^n$, if k is a positive integer such that for all $t \in I$, $\{\gamma'(t), \ldots, \gamma^{(k)}(t)\}$ is a linearly independent set, we say that the curve $\gamma: I \to \mathbb{R}^n$ is regular of order k.

By definition, for any $t \in I$, $\gamma'(t)$ is a nonzero vector. Thus, $\{\gamma'(t)\}$ is a linearly independent set. Hence, a curve $\gamma: I \to \mathbb{R}^n$ must be regular of order 1. If a curve is regular of order k, then $1 \le k \le n$.

Now we define the Frenet frame for a curve.

Definition 2.12 (The Frenet Frame). Let $n \geq 2$, and let $\gamma : I \to \mathbb{R}^n$ be a curve in \mathbb{R}^n that has order n-1. For any $t \in I$, the Frenet frame $\{\mathbf{T}(t), \mathbf{N}_1(t), \dots, \mathbf{N}_{n-1}(t)\}$ is an orthonormal set, where the vectors $\mathbf{T}(t), \mathbf{N}_1(t), \dots, \mathbf{N}_{n-2}(t)$ are obtained by applying the Gram-Schmidt process to the linearly independent set $\{\gamma'(t), \dots, \gamma^{(n-1)}(t)\}$, and the vector $\mathbf{N}_{n-1}(t)$ is defined by the cross product

$$\mathbf{N}_{n-1}(t) = \mathscr{P}(\mathbf{T}(t), \mathbf{N}_1(t), \dots, \mathbf{N}_{n-2}(t)).$$

The frame matrix $\mathbf{F}(t)$ is defined as the $n \times n$ matrix

$$\mathbf{F}(t) = \left[\mathbf{T}(t) \mid \mathbf{N}_1(t) \mid \cdots \mid \mathbf{N}_{n-1}(t) \right].$$

By Corollary 2.8, for any $t \in I$, the Frenet frame $\{\mathbf{T}(t), \mathbf{N}_1(t), \dots, \mathbf{N}_{n-1}(t)\}$ is a positively oriented orthonormal basis of \mathbb{R}^n . Equivalently, the frame matrix $\mathbf{F}(t)$ is an orthogonal matrix with determinant 1. The algorithm of the Gram-Schmidt process and the definition of the generalized cross product show that each of the vectors $\mathbf{T}(t), \mathbf{N}_1(t), \dots, \mathbf{N}_{n-1}(t)$ is a smooth function of t.

Our definition differs from some literatures which only consider curves of order n in \mathbb{R}^n . They define the Frenet frames as the orthonormal set obtained by applying the Gram-Schmidt process to the set $\{\gamma'(t), \gamma''(t), \ldots, \gamma^{(n)}(t)\}$. The definition we use here produces the same vectors $\mathbf{T}(t), \ldots, \mathbf{N}_{n-2}(t)$, and produces the vector $\mathbf{N}_{n-1}(t)$ that might differ by a sign.

One can show that the Frenet frame is independent of parametrizations.

Given a curve $\gamma: I \to \mathbb{R}^n$ of order n-1, let $\mathbf{A}(t)$ and $\mathbf{F}(t)$ be respectively the canonical matrix and the frame matrix of γ . Then there exists an $n \times n$ matrix $\mathbf{R}(t) = [R_{i,j}(t)]$ such that

$$\mathbf{A}(t) = \mathbf{F}(t)\mathbf{R}(t).$$

Since the first n-1 column vectors of $\mathbf{F}(t)$ are obtained by applying the Gram-Schmidt process to the first n-1 column vectors of $\mathbf{A}(t)$, $\mathbf{R}(t)$ is an upper triangular matrix. Moreover, for $1 \le j \le n-1$, the diagonal entry $R_{j,j}(t)$ is positive.

Since $\mathbf{F}(t)$ is an orthogonal matrix and $\mathbf{R}(t)$ is an upper triangular matrix, $\mathbf{A}(t) = \mathbf{F}(t)\mathbf{R}(t)$ gives a QR-decomposition of the matrix $\mathbf{A}(t)$. This explains our choice of the notation for $\mathbf{R}(t)$.

Now we define the generalized curvatures κ_i , $1 \le i \le n-1$.

Definition 2.13 (Generalized Curvatures). Let $\gamma: I \to \mathbb{R}^n$ be an arclength parametrized curve of order n-1. For $s \in I$, let $\{\mathbf{T}(s), \mathbf{N}_1(s), \dots, \mathbf{N}_{n-1}(s)\}$ be the Frenet frame of the curve at $\gamma(s)$. The first curvature $\kappa_1(s)$ is defined as

$$\kappa_1(s) = \langle \mathbf{T}'(s), \mathbf{N}_1(s) \rangle.$$
(2.1)

For $2 \le i \le n-1$, the *i*-th curvature $\kappa_i(s)$ is defined as

$$\kappa_i(s) = \langle \mathbf{N}'_{i-1}(s), \mathbf{N}_i(s) \rangle. \tag{2.2}$$

If a curve is not parametrized by arclength, we define the generalized curvatures by its arclength reparametrization.

The Frenet-Serret formulas is a set of equations that express the derivatives of $\mathbf{T}(s)$, $\mathbf{N}_1(s)$, ..., $\mathbf{N}_{n-1}(s)$ with respect to s in terms of $\mathbf{T}(s)$, $\mathbf{N}_1(s)$, ..., $\mathbf{N}_{n-1}(s)$.

Theorem 2.14 (Frenet-Serret Formulas). Let $\gamma: I \to \mathbb{R}^n$ be a curve of order n-1 that is parametrized by arclength. For $s \in I$, let $\{\mathbf{T}(s), \mathbf{N}_1(s), \dots, \mathbf{N}_{n-1}(s)\}$ be the Frenet frame at $\gamma(s)$, and let $\kappa_i(s)$, $1 \le i \le n-1$ be the generalized curvatures. Then

$$\frac{d\mathbf{T}}{ds} = \kappa_1 \mathbf{N}_1,$$

$$\frac{d\mathbf{N}_1}{ds} = -\kappa_1 \mathbf{T} + \kappa_2 \mathbf{N}_2,$$

$$\vdots$$

$$\frac{d\mathbf{N}_j}{ds} = -\kappa_j \mathbf{N}_{j-1} + \kappa_{j+1} \mathbf{N}_{j+1}, \qquad 2 \le j \le n-2,$$

$$\vdots$$

$$\frac{d\mathbf{N}_{n-1}}{ds} = -\kappa_{n-1} \mathbf{N}_{n-2}.$$

In terms of the frame matrix $\mathbf{F}(s) = \left[\mathbf{T}(s) \mid \mathbf{N}_1(s) \mid \cdots \mid \mathbf{N}_{n-1}(s)\right]$, these Frenet-Serret formulas can be written as

$$\frac{d\mathbf{F}(s)}{ds} = \mathbf{F}(s)\mathbf{C}(s),$$

where C(s) is the anti-symmetric matrix

$$\mathbf{C}(s) = \begin{bmatrix} 0 & -\kappa_1(s) & 0 & \cdots & 0 & 0\\ \kappa_1(s) & 0 & -\kappa_2(s) & \cdots & 0 & 0\\ 0 & \kappa_2(s) & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & -\kappa_{n-1}(s)\\ 0 & 0 & 0 & \cdots & \kappa_{n-1}(s) & 0 \end{bmatrix}.$$

3. GENERALIZED CURVATURES

For a curve $\gamma: I \to \mathbb{R}^n$ of order n-1, we have defined its generalized curvatures $\kappa_1, \ldots, \kappa_{n-1}$. When $n=3, \kappa_1$ is the curvature κ, κ_2 is the torsion τ . For general n, it has been proved that the curvatures $\kappa_1, \ldots, \kappa_{n-2}$ are positive-valued (see for example, [Glu66, Glu67]). Here we prove this fact in a slightly different way, without the need to define the excess vectors as in [Glu66]. We first prove the following theorem.

Theorem 3.1. Given that $\gamma: I \to \mathbb{R}^n$ is an aclength parametrized curve of order n-1, let $\kappa_1, \ldots, \kappa_{n-1}$ be the generalized curvatures of $\gamma: I \to \mathbb{R}^n$. Denote by $\mathbf{A}(s)$ and $\mathbf{F}(s)$ the canonical matrix and the frame matrix of γ . Let $\mathbf{R}(s) = [R_{i,j}(s)]$ be the $n \times n$ upper triangular matrix so that $\mathbf{A}(s) = \mathbf{F}(s)\mathbf{R}(s)$. Then $R_{1,1}(s) = 1$, and for $1 \leq i \leq n$, the i-th diagonal entry $1 \leq i \leq n$ is given by

$$R_{j,j}(s) = \prod_{i=1}^{j-1} \kappa_i(s) = \kappa_1(s)\kappa_2(s)\cdots\kappa_{j-1}(s).$$

Proof. By definition, for $1 \le j \le n$, we have

$$\gamma^{(j)}(s) = R_{1,j}(s)\mathbf{T}(s) + \sum_{i=2}^{j} R_{i,j}(s)\mathbf{N}_{i-1}(s).$$
(3.1)

In particular,

$$\gamma'(s) = R_{1,1}(s)\mathbf{T}(s).$$

Since $T(s) = \gamma'(s)$, we find that

$$R_{1,1}(s) = 1.$$

For $2 \le j \le n$, (3.1) gives

$$R_{j,j}(s) = \langle \boldsymbol{\gamma}^{(j)}(s), \mathbf{N}_{j-1}(s) \rangle.$$

Using the fact that $\gamma'(s) = \mathbf{T}(s)$, we have

$$R_{2,2}(s) = \langle \boldsymbol{\gamma}''(s), \mathbf{N}_1(s) \rangle = \langle \mathbf{T}'(s), \mathbf{N}_1(s) \rangle = \kappa_1(s).$$

Now given $2 \le j \le n-1$, assume that we have shown that

$$R_{i,j}(s) = \kappa_1(s) \dots \kappa_{j-1}(s). \tag{3.2}$$

Differentiating (3.1) with respect to s, we have

$$\boldsymbol{\gamma}^{(j+1)}(s) = R'_{1,j}(s)\mathbf{T}(s) + \sum_{i=2}^{j} R'_{i,j}(s)\mathbf{N}_{i-1}(s) + R_{1,j}(s)\mathbf{T}'(s) + \sum_{i=2}^{j} R_{i,j}(s)\mathbf{N}'_{i-1}(s).$$

It follows that

$$R_{j+1,j+1}(s) = \langle \boldsymbol{\gamma}^{(j+1)}(s), \mathbf{N}_{j}(s) \rangle$$
$$= R_{1,j}(s) \langle \mathbf{T}'(s), \mathbf{N}_{j}(s) \rangle + \sum_{i=1}^{j} R_{i,j}(s) \langle \mathbf{N}'_{i-1}(s), \mathbf{N}_{j}(s) \rangle.$$

Since $j \geq 2$, $\langle \mathbf{T}'(s), \mathbf{N}_j(s) \rangle = 0$. On the other hand, $\langle \mathbf{N}'_{i-1}(s), \mathbf{N}_j(s) \rangle = 0$ for $2 \leq i \leq j-1$. Therefore,

$$R_{j+1,j+1}(s) = R_{j,j}(s) \langle \mathbf{N}'_{j-1}(s), \mathbf{N}_j(s) \rangle = R_{j,j}(s) \kappa_j(s).$$

By the inductive hypothesis (3.2), we conclude that

$$R_{j+1,j+1}(s) = \kappa_1(s) \dots \kappa_{j-1}(s) \kappa_j(s).$$

This completes the proof by induction.

An immediate consequence of Theorem 3.1 is the following result proved in [Glu66].

Corollary 3.2. Given that $\gamma: I \to \mathbb{R}^n$ is an aclength parametrized curve of order n-1, let $\kappa_1, \ldots, \kappa_{n-1}$ be the generalized curvatures of $\gamma: I \to \mathbb{R}^n$. Denote by $\mathbf{A}(s)$ and $\mathbf{F}(s)$ the canonical matrix and the frame matrix of γ . Let $\mathbf{R}(s) = [R_{i,j}(s)]$ be the $n \times n$ upper triangular matrix so that $\mathbf{A}(s) = \mathbf{F}(s)\mathbf{R}(s)$. Then for $1 \le i \le n-1$,

$$\kappa_i(s) = \frac{R_{i+1,i+1}(s)}{R_{i,i}(s)}. (3.3)$$

Since $R_{j,j}(s) > 0$ for $1 \le j \le n-1$, we obtain the following immediately.

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Theorem 3.3. Let $n \geq 3$, and let $\gamma : I \to \mathbb{R}^n$ be a parametrized curve of order n-1. Then the generalized curvatures $\kappa_1, \ldots, \kappa_{n-2}$ are positive-valued.

Remark 3.4. It can be shown that $R_i(s)$ is the norm of the excess vector $E_i(s)$ defined in [Glu66]. Hence, the formula (3.3) is essentially Theorem 3.1 in [Glu66].

For the sign of the generalized curvature κ_{n-1} , we have the following.

Theorem 3.5. Given that $\gamma: I \to \mathbb{R}^n$ is a parametrized curve of order n-1, let $\mathbf{A}(t)$ be the canonical matrix of γ , and let $\kappa_{n-1}(t)$ be the (n-1)-th generalized curvature of the curve at $\gamma(t)$. Then $\kappa_{n-1}(t)$ has the same sign as $\det \mathbf{A}(t)$. In particular, $\kappa_{n-1}(t)=0$ if and only if $\{\gamma'(t),\ldots,\gamma^{(n)}(t)\}$ is a linearly dependent set.

Proof. It is sufficient to prove this theorem under the assumption that $\gamma: I \to \mathbb{R}^n$ is an arclength parametrization. Let $\mathbf{F}(s)$ be the frame matrix of γ , and let $\mathbf{R}(s)$ be the $n \times n$ upper triangular matrix such that

$$\mathbf{A}(s) = \mathbf{F}(s)\mathbf{R}(s).$$

Since $\det \mathbf{F}(s) = 1$,

$$\det \mathbf{A}(s) = \det \mathbf{R}(s) = \prod_{i=1}^{n} R_{i,i}(s).$$

Since $R_{i,i}(s) > 0$ for $1 \le i \le n-1$, $\det \mathbf{A}(s)$ has the same sign as $R_{n,n}(s)$. By Theorem 3.1,

$$R_{n,n}(s) = \kappa_1(s) \dots \kappa_{n-2}(s) \kappa_{n-1}(s).$$

By Theorem 3.3, $\kappa_i(s) > 0$ if $1 \le i \le n-2$. Hence, $R_{n,n}(s)$ has the same sign as $\kappa_{n-1}(s)$. This implies that $\kappa_{n-1}(s)$ has the same sign as $\det \mathbf{A}(s)$. In particular, $\kappa_{n-1}(s) = 0$ if and only if $\det \mathbf{A}(s) = 0$, if and only if $\{\gamma'(s), \ldots, \gamma^{(n)}(s)\}$ is a linearly dependent set.

To find the generalized curvatures of a curve $\gamma: I \to \mathbb{R}^n$ of order n-1 that is parametrized by arclength, we can apply the Gram-Schmidt process to the set $\{\gamma'(s), \ldots, \gamma^{(n-1)}(s)\}$. In the process, we can compute the coefficients $R_{i,i}(s) = \langle \gamma^{(i)}(s), \mathbf{N}_{i-1}(s) \rangle$ for $2 \le i \le n-1$. The coefficient $R_{n,n}(s) = \langle \gamma^{(n)}(s), \mathbf{N}_{n-1}(s) \rangle$ can then be computed using the fact that

$$\prod_{i=1}^{n} R_{i,i} = \det \mathbf{A}(s).$$

This allows us to compute the generalized curvatures $\kappa_1(s), \ldots, \kappa_{n-1}(s)$ using Corollary 3.2.

Although in principle one can always reparametrize a curve $\gamma:I\to\mathbb{R}^n$ by arclength, this is by no means a simple task. It is desirable to have formulas for the generalized curvatures of a parametrized curve $\gamma:I\to\mathbb{R}^n$ of order n-1 purely in terms of $\gamma'(t),\ldots,\gamma^{(n)}(t)$, generalizing the n=3 formulas

$$\kappa_1(t) = \frac{\|\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)\|}{\|\boldsymbol{\gamma}'(t)\|^3}, \qquad \kappa_2(t) = \frac{\langle \boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t), \boldsymbol{\gamma}'''(t) \rangle}{\|\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)\|^2}.$$
 (3.4)

Note that

$$\|\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)\|^{2} = \det \begin{bmatrix} \langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}'(t) \rangle & \langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}''(t) \rangle \\ \langle \boldsymbol{\gamma}''(t), \boldsymbol{\gamma}'(t) \rangle & \langle \boldsymbol{\gamma}''(t), \boldsymbol{\gamma}''(t) \rangle \end{bmatrix},$$
$$\langle \boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t), \boldsymbol{\gamma}'''(t) \rangle = \det \begin{bmatrix} \boldsymbol{\gamma}'(t) \mid \boldsymbol{\gamma}''(t) \mid \boldsymbol{\gamma}'''(t) \end{bmatrix}.$$

To shed further lights, note that

$$\left(\det\left[\boldsymbol{\gamma}'(t)\mid\boldsymbol{\gamma}''(t)\mid\boldsymbol{\gamma}'''(t)\right]\right)^{2} = \det\left[\frac{\boldsymbol{\gamma}'(t)^{T}}{\boldsymbol{\gamma}'''(t)^{T}}\right] \left[\boldsymbol{\gamma}'(t)\mid\boldsymbol{\gamma}''(t)\mid\boldsymbol{\gamma}'''(t)\right] \\
= \det\left[\frac{\langle\boldsymbol{\gamma}'(t),\boldsymbol{\gamma}'(t)\rangle \quad \langle\boldsymbol{\gamma}'(t),\boldsymbol{\gamma}''(t)\rangle \quad \langle\boldsymbol{\gamma}'(t),\boldsymbol{\gamma}'''(t)\rangle}{\langle\boldsymbol{\gamma}''(t),\boldsymbol{\gamma}''(t)\rangle \quad \langle\boldsymbol{\gamma}''(t),\boldsymbol{\gamma}'''(t)\rangle}\right] \\
= \det\left[\frac{\langle\boldsymbol{\gamma}'(t),\boldsymbol{\gamma}'(t)\rangle \quad \langle\boldsymbol{\gamma}''(t),\boldsymbol{\gamma}''(t)\rangle \quad \langle\boldsymbol{\gamma}''(t),\boldsymbol{\gamma}'''(t)\rangle}{\langle\boldsymbol{\gamma}'''(t),\boldsymbol{\gamma}''(t)\rangle \quad \langle\boldsymbol{\gamma}'''(t),\boldsymbol{\gamma}'''(t)\rangle}\right].$$

Motivated by this, for any parametrized curve $\gamma: I \to \mathbb{R}^n$, let $\mathbf{A}(t)$ be its canonical matrix, and consider the matrix

$$\mathbf{B}(t) = \mathbf{A}(t)^T \mathbf{A}(t).$$

The (i, j)-component of $\mathbf{B}(t)$ is

$$B_{i,j}(t) = \langle \boldsymbol{\gamma}^{(i)}(t), \boldsymbol{\gamma}^{(j)}(t) \rangle.$$

For $1 \le i \le n$, let $\mathbf{M}_i(t)$ be the $i \times i$ matrix

$$\mathbf{M}_i(t) = egin{bmatrix} \langle oldsymbol{\gamma}'(t), oldsymbol{\gamma}'(t)
angle & \cdots & \langle oldsymbol{\gamma}'(t), oldsymbol{\gamma}^{(i)}(t)
angle \ dots & \ddots & dots \ \langle oldsymbol{\gamma}^{(i)}(t), oldsymbol{\gamma}'(t)
angle & \cdots & \langle oldsymbol{\gamma}^{(i)}(t), oldsymbol{\gamma}^{(i)}(t)
angle \end{bmatrix},$$

which consists of the first i rows and first i columns of $\mathbf{B}(t)$. The determinant of $\mathbf{M}_i(t)$ is called the i-th leading principal minor of $\mathbf{B}(t)$.

Our main result is that the generalized curvatures $\kappa_1(t), \ldots, \kappa_{n-1}(t)$ can be expressed in terms of the determinants of $\mathbf{M}_i(t)$. If $\gamma: I \to \mathbb{R}^{n-1}$ is a curve of order n-1, for $1 \le i \le n-1$, since $\{\gamma'(t), \ldots, \gamma^i(t)\}$ is a linearly independent set. This implies that the matrix

$$\mathbf{M}_i(t) = \left[\boldsymbol{\gamma}'(t) \mid \cdots \mid \boldsymbol{\gamma}^{(i)}(t) \right]^T \left[\boldsymbol{\gamma}'(t) \mid \cdots \mid \boldsymbol{\gamma}^{(i)}(t) \right]$$

is positive definite, and so $\det \mathbf{M}_i(t) > 0$. In fact, $\det \mathbf{M}_i(t)$ is the square of the volume of the parallelepiped spanned by $\gamma'(t), \ldots, \gamma^{(i)}(t)$.

Theorem 3.6. Let $n \geq 2$. Given a parametrized curve $\gamma : I \to \mathbb{R}^n$ of order n-1, let $\mathbf{A}(t)$ be its canonical matrix, and let $\kappa_1(t), \ldots, \kappa_{n-1}(t)$ be the generalized curvatures. For $1 \leq i \leq n$, denote by $\mathbf{M}_i(t)$ the matrix

$$\mathbf{M}_i(t) = egin{bmatrix} \langle oldsymbol{\gamma}'(t), oldsymbol{\gamma}'(t)
angle & \cdots & \langle oldsymbol{\gamma}'(t), oldsymbol{\gamma}^{(i)}(t)
angle \ dots & \ddots & dots \ \langle oldsymbol{\gamma}^{(i)}(t), oldsymbol{\gamma}'(t)
angle & \cdots & \langle oldsymbol{\gamma}^{(i)}(t), oldsymbol{\gamma}^{(i)}(t)
angle \end{bmatrix}.$$

Then for $n \geq 3$,

$$\kappa_1(t) = \frac{\sqrt{\det \mathbf{M}_2(t)}}{\|\gamma'(t)\|^3};$$

for $2 \le i \le n-2$,

$$\kappa_i(t) = \frac{\sqrt{\det \mathbf{M}_{i+1}(t) \det \mathbf{M}_{i-1}(t)}}{\|\boldsymbol{\gamma}'(t)\| \det \mathbf{M}_i(t)};$$

and for $n \geq 2$,

$$\kappa_{n-1}(t) = \frac{\det \mathbf{A}(t)}{\|\boldsymbol{\gamma}'(t)\| \det \mathbf{M}_{n-1}(t)} \sqrt{\det \mathbf{M}_{n-2}(t)}.$$

Proof. Let $s: I \to J$ be an arclength function of $\gamma: I \to \mathbb{R}^n$. Then

$$s'(t) = \|\boldsymbol{\gamma}'(t)\|.$$

Let $\widetilde{\gamma}:J\to\mathbb{R}^n$ be the arclength reparametrization of $\gamma:I\to\mathbb{R}^n$ so that

$$\gamma(t) = \widetilde{\gamma}(s(t)).$$

Denote by $\widetilde{\mathbf{A}}(s)$ the canonical matrix of $\widetilde{\gamma}(s)$. By Proposition 2.10, there exists an upper triangular matrix $\mathbf{U}(t) = [U_{i,j}(t)]$ with diagonal entries $U_{j,j}(t) = s'(t)^j$, $1 \le j \le n$, such that

$$\mathbf{A}(t) = \widetilde{\mathbf{A}}(s(t))\mathbf{U}(t).$$

Let $\{\mathbf{T}(t), \mathbf{N}_1(s), \dots, \mathbf{N}_{n-1}(t)\}$ be the Frenet frame at $\gamma(t) = \widetilde{\gamma}(s(t))$, and let $\mathbf{F}(t) = \widetilde{\mathbf{F}}(s(t))$ be the corresponding frame matrix. Then there are upper triangular matrices $\mathbf{R}(t) = [R_{i,j}(t)]$ and $\widetilde{\mathbf{R}}(s) = [\widetilde{R}_{i,j}(s)]$ such that

$$\mathbf{A}(t) = \mathbf{F}(t)\mathbf{R}(t)$$
 and $\widetilde{\mathbf{A}}(s) = \widetilde{\mathbf{F}}(s)\widetilde{\mathbf{R}}(s)$.

It follows that

$$\mathbf{R}(t) = \widetilde{\mathbf{R}}(s(t))\mathbf{U}(t).$$

Since the matrices $\widetilde{\mathbf{R}}(s)$ and $\mathbf{U}(t)$ are upper triangular, we find that for $1 \leq j \leq n$,

$$R_{j,j}(t) = \widetilde{R}_{j,j}(s(t))U_{j,j}(t) = \widetilde{R}_{j,j}(s(t))s'(t)^{j}.$$

When j=1, Theorem 3.1 says that $\widetilde{R}_{1,1}(s)=1$. Therefore, we have

$$R_{1,1}(t) = s'(t).$$

For $1 \le i \le n-1$, Corollary 3.2 says that

$$\kappa_i(t) = \frac{\widetilde{R}_{i+1,i+1}(s(t))}{\widetilde{R}_{i,i}(s(t))} = \frac{1}{s'(t)} \frac{R_{i+1,i+1}(t)}{R_{i,i}(t)}.$$
(3.5)

Now consider the matrix $\mathbf{B}(t) = \mathbf{A}(t)^T \mathbf{A}(t)$. Since $\mathbf{A}(t) = \mathbf{F}(t)\mathbf{R}(t)$ and $\mathbf{F}(t)$ is an orthogonal matrix, we have

$$\mathbf{B}(t) = \mathbf{R}(t)^T \mathbf{F}(t)^T \mathbf{F}(t) \mathbf{R}(t) = \mathbf{R}(t)^T \mathbf{R}(t).$$

For fixed $1 \le i \le n$, we partition the matrix $\mathbf{R}(t)$ into 4 blocks

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{V}_{i,1}(t) & \mathbf{V}_{i,2}(t) \\ \mathbf{V}_{i,3}(t) & \mathbf{V}_{i,4}(t) \end{bmatrix},$$

such that $V_{i,1}$ is an $i \times i$ matrix. Since $\mathbf{R}(t)$ is upper triangular, $V_{i,1}(t)$ is an upper triangular matrix and $V_{i,3}(t) = \mathbf{0}$ is the zero matrix. Therefore,

$$\mathbf{B}(t) = \left[egin{array}{c|c|c} \mathbf{V}_{i,1}(t)^T & \mathbf{0} \\ \hline \mathbf{V}_{i,2}(t)^T & \mathbf{V}_{i,4}(t)^T \end{array}
ight] \left[egin{array}{c|c|c} \mathbf{V}_{i,1}(t) & \mathbf{V}_{i,2}(t) \\ \hline \mathbf{0} & \mathbf{V}_{i,4}(t) \end{array}
ight],$$

and we obtain

$$\mathbf{M}_i(t) = \mathbf{V}_{i,1}(t)^T \mathbf{V}_{i,1}(t).$$

It follows that

$$\det \mathbf{M}_i(t) = (\det \mathbf{V}_{i,1}(t))^2.$$

By definition, $V_{i,1}(t)$ is an upper triangular $i \times i$ matrix with diagonal entries $R_{1,1}(t), \dots, R_{i,i}(t)$. Hence,

$$\det \mathbf{M}_i(t) = \left(\prod_{j=1}^i R_{j,j}(t)\right)^2.$$

This gives

$$\det \mathbf{M}_1(t) = R_{1,1}(t)^2 = s'(t)^2,$$

and for $1 \le i \le n-1$,

$$R_{i+1,i+1}(t)^{2} = \frac{\det \mathbf{M}_{i+1}(t)}{\det \mathbf{M}_{i}(t)}.$$
(3.6)

Using the fact that $s'(t) = ||\gamma'(t)||$, (3.5) and (3.6) give

$$\kappa_1(t)^2 = \frac{1}{s'(t)^2} \frac{R_{2,2}(t)^2}{R_{1,1}(t)^2} = \frac{1}{\|\boldsymbol{\gamma}'(t)\|^4} \frac{\det \mathbf{M}_2(t)}{\det \mathbf{M}_1(t)} = \frac{\det \mathbf{M}_2(t)}{\|\boldsymbol{\gamma}'(t)\|^6}.$$

Theorem 3.3 says that if $n \ge 3$, $\kappa_1(t) > 0$. Hence, if $n \ge 3$,

$$\kappa_1(t) = \frac{\sqrt{\det \mathbf{M}_2(t)}}{\|\boldsymbol{\gamma}'(t)\|^3}.$$

If $2 \le i \le n - 1$, (3.5) and (3.6) give

$$\kappa_i(t)^2 = \frac{1}{\|\gamma'(t)\|^2} \frac{\det \mathbf{M}_{i+1}(t) \det \mathbf{M}_{i-1}(t)}{[\det \mathbf{M}_i(t)]^2}.$$
(3.7)

If $2 \le i \le n-2$, Theorem 3.3 says that $\kappa_i(t) > 0$. Therefore, when $2 \le i \le n-2$,

$$\kappa_i(t) = \frac{\sqrt{\det \mathbf{M}_{i+1}(t) \det \mathbf{M}_{i-1}(t)}}{\|\boldsymbol{\gamma}'(t)\| \det \mathbf{M}_i(t)}.$$

Finally, we notice that

$$\det \mathbf{M}_n(t) = \det \mathbf{B}(t) = (\det \mathbf{A}(t))^2.$$

Therefore, (3.7) gives

$$\kappa_{n-1}(t)^2 = \frac{1}{\|\boldsymbol{\gamma}'(t)\|^2} \frac{[\det \mathbf{A}(t)]^2 \det \mathbf{M}_{n-2}(t)}{[\det \mathbf{M}_{n-1}(t)]^2}.$$

By Theorem 3.5, $\kappa_{n-1}(t)$ has the same sign as det $\mathbf{A}(t)$. It follows that

$$\kappa_{n-1}(t) = \frac{\det \mathbf{A}(t)}{\|\boldsymbol{\gamma}'(t)\| \det \mathbf{M}_{n-1}(t)} \sqrt{\det \mathbf{M}_{n-2}(t)}.$$

Note that since $\det \mathbf{M}_1(t) = \|\boldsymbol{\gamma}'(t)\|^2$, if $n \geq 4$, we can simplify $\kappa_2(t)$ to

$$\kappa_2(t) = \frac{\sqrt{\det \mathbf{M}_3(t)}}{\det \mathbf{M}_2(t)}.$$

If n = 3, we have the classical formula

$$\kappa_2(t) = \frac{\det \mathbf{A}(t)}{\det \mathbf{M}_2(t)}.$$

A disguised form of the results in Theorem 3.6 have been obtained in [Ger62, Glu66, Gut11], where $\sqrt{\det \mathbf{M}_i(t)}$ is written as the volume of the parallelepiped spanned by $\gamma'(t), \ldots, \gamma^{(i)}(t)$. In fact, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a linearly independent set in \mathbb{R}^n , $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is the orthonormal set obtained by applying the Gram-Schmidt process to $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$, and

$$R_{i,j} = \langle \mathbf{u}_i, \mathbf{v}_i \rangle,$$

then $[R_{i,j}]$ is an upper triangular matrix with positive diagonal entries $R_{j,j}$, $1 \le j \le r$. Moreover, for $1 \le j \le r$,

$$\mathbf{v}_{j} = \sum_{i=1}^{j} R_{i,j} \mathbf{u}_{i} = \sum_{i=1}^{j-1} R_{i,j} \mathbf{u}_{i} + R_{j,j} \mathbf{u}_{j}.$$

This implies that

$$\mathbf{w}_j = \mathbf{v}_j - \sum_{i=1}^{j-1} \langle \mathbf{u}_i, \mathbf{v}_j \rangle \mathbf{u}_i = R_{j,j} \mathbf{u}_j.$$

Since \mathbf{w}_j is the component of \mathbf{v}_j perpendicular to the subspace spanned by $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$, we find that the volume of the parallelepiped spanned by $\mathbf{v}_1, \dots, \mathbf{v}_r$ is

$$\|\mathbf{w}_1\|\cdots\|\mathbf{w}_r\|=R_{1,1}\cdots R_{r,r}.$$

Theorem 3.6 provides an efficient way to compute the generalized curvatures of a curve $\gamma: I \to \mathbb{R}^n$ under any parametrization. One first computes the canonical matrix $\mathbf{A}(t)$, and then the matrix $\mathbf{B}(t) = \mathbf{A}(t)^T \mathbf{A}(t)$. From this, one can extract the matrices $\mathbf{M}_i(t)$ and compute the curvatures $\kappa_1(t), \ldots, \kappa_{n-1}(t)$ using their determinants and the formulas given in Theorem 3.6. In the process, there is no need to apply the Gram-Schmidt algorithm.

As an example, let us consider the curve in \mathbb{R}^4 given by $\gamma: \mathbb{R} \to \mathbb{R}^4$,

$$\boldsymbol{\gamma}(t) = (t, t^2, t^3, t^4).$$

Using any computer algebra that can perform symbolic computation, we can easily find that the canonical matrix of $\gamma: I \to \mathbb{R}^4$ is

$$\mathbf{A}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2t & 2 & 0 & 0 \\ 3t^2 & 6t & 6 & 0 \\ 4t^3 & 12t^2 & 24t & 24 \end{bmatrix}.$$

It follows that

$$\mathbf{B}(t) = \mathbf{A}(t)^{T} \mathbf{A}(t) = \begin{bmatrix} 1 + 4t^{2} + 9t^{4} + 16t^{6} & 4t + 18t^{3} + 48t^{5} & 18t^{2} + 96t^{4} & 96t^{3} \\ 4t + 18t^{3} + 48t^{5} & 4 + 36t^{2} + 144t^{4} & 36t + 288t^{3} & 288t^{2} \\ 18t^{2} + 96t^{4} & 36t + 288t^{3} & 36 + 576t^{2} & 576t \\ 96t^{3} & 288t^{2} & 576t & 576 \end{bmatrix}.$$

From this, we obtain the first 3 leading principal minors of $\mathbf{B}(t)$, and the determinant of $\mathbf{A}(t)$ as

$$\det \mathbf{M}_1(t) = 1 + 4t^2 + 9t^4 + 16t^6 = \|\boldsymbol{\gamma}'(t)\|^2,$$

$$\det \mathbf{M}_2(t) = 4 + 36t^2 + 180t^4 + 256t^6 + 144t^8,$$

$$\det \mathbf{M}_3(t) = 144 + 2304t^2 + 5184t^4 + 2304t^6,$$

$$\det \mathbf{A}(t) = 288.$$

By Theorem 3.6, the generalized curvatures $\kappa_1(t)$, $\kappa_2(t)$, $\kappa_3(t)$ are given by

$$\kappa_{1}(t) = \frac{\sqrt{\det \mathbf{M}_{2}(t)}}{\|\boldsymbol{\gamma}'(t)\|^{3}} = \frac{\sqrt{4 + 36t^{2} + 180t^{4} + 256t^{6} + 144t^{8}}}{(1 + 4t^{2} + 9t^{4} + 16t^{6})^{\frac{3}{2}}},$$

$$\kappa_{2}(t) = \frac{\sqrt{\det \mathbf{M}_{3}(t)}}{\det \mathbf{M}_{2}(t)} = \frac{\sqrt{144 + 2304t^{2} + 5184t^{4} + 2304t^{6}}}{4 + 36t^{2} + 180t^{4} + 256t^{6} + 144t^{8}},$$

$$\kappa_{3}(t) = \frac{\det \mathbf{A}(t)\sqrt{\det \mathbf{M}_{2}(t)}}{\|\boldsymbol{\gamma}'(t)\|\det \mathbf{M}_{3}(t)}$$

$$= \frac{288}{144 + 2304t^{2} + 5184t^{4} + 2304t^{6}}\sqrt{\frac{4 + 36t^{2} + 180t^{4} + 256t^{6} + 144t^{8}}{1 + 4t^{2} + 9t^{4} + 16t^{6}}}.$$

4. GENERALIZATIONS

In this section, we consider the general case of a curve $\gamma: I \to \mathbb{R}^n$ which does not necessary have order n-1.

Given a curve $\gamma: I \to \mathbb{R}^n$, we define the canonical matrix $\mathbf{A}(t)$ and consider the matrix $\mathbf{B}(t) = \mathbf{A}(t)^T \mathbf{A}(t)$. One can then compute $\det \mathbf{A}(t)$ and the leading principal minors of $\mathbf{B}(t)$ given by

$$\det \mathbf{M}_{i}(t) = \det \begin{bmatrix} \langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}'(t) \rangle & \cdots & \langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}^{(i)}(t) \rangle \\ \vdots & \ddots & \vdots \\ \langle \boldsymbol{\gamma}^{(i)}(t), \boldsymbol{\gamma}'(t) \rangle & \cdots & \langle \boldsymbol{\gamma}^{(i)}(t), \boldsymbol{\gamma}^{(i)}(t) \rangle \end{bmatrix}, \qquad 1 \leq i \leq n.$$

For $1 \le i \le n$, define the set I_i as

$$I_i = \{t \in I | \det \mathbf{M}_i(t) = 0\}.$$

Since $\det \mathbf{M}_i(t)$ is a continuous function, I_i must be a closed subset of I. Note that $t \in I_i$ if and only if the set $\{\gamma'(t), \ldots, \gamma^{(i)}(t)\}$ is linearly dependent. Hence, $I_1 = \emptyset$, and

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset I$$
.

Since $\{\gamma'(t), \ldots, \gamma^{(n+1)}(t)\}$ must be a linearly dependent set, we extrapolate and define $I_{n+1} = I$.

For any $1 \le k \le n$, the curve has order k if and only if $I_k = \emptyset$.

Since $I_1=\emptyset$ and $I_{n+1}=I$, there exists $1\leq r\leq n$ such that $I_{r+1}=I$ and $I_r\neq I$, then $\widetilde{I}_r=I\setminus I_r$ is an open nonempty subset of real numbers. It can be written as a disjoint union of countably many open intervals. We can restrict the curve γ to each of these open intervals and consider them separately. Thus, it is sufficient to consider a curve $\gamma:I\to\mathbb{R}^n$ so that for all $t\in I, \gamma'(t),\ldots,\gamma^{(r)}(t)$ are linearly independent, but $\gamma'(t),\ldots,\gamma^{(r)}(t),\gamma^{(r+1)}(t)$ are linearly dependent.

If r = n - 1 or r = n, we can define the Frenet frame as before.

If $1 \le r \le n-2$, we can still define the orthonormal set $\{\mathbf{T}(t), \mathbf{N}_1(t), \dots, \mathbf{N}_{r-1}(t)\}$ by applying the Gram-Schmidt process to the linearly independent set $\gamma'(t), \dots, \gamma^{(r)}(t)$. Then one can define the curvatures $\kappa_1(t), \dots, \kappa_{r-2}(t)$ as before. They are positive valued.

Since $\gamma'(t), \ldots, \gamma^{(r)}(t)$ are linearly independent, but $\gamma'(t), \ldots, \gamma^{(r)}(t), \gamma^{(r+1)}(t)$ are linearly dependent, we find that

$$\boldsymbol{\gamma}^{(r+1)}(t) \in \operatorname{span}\{\boldsymbol{\gamma}'(t), \dots, \boldsymbol{\gamma}^{(r)}(t)\} = \operatorname{span}\{\mathbf{T}(t), \mathbf{N}_1(t), \dots, \mathbf{N}_{r-1}(t)\}.$$

This implies that

$$\mathbf{N}_{r-1}(t) \in \operatorname{span}\{\boldsymbol{\gamma}'(t), \dots, \boldsymbol{\gamma}^{(r)}(t)\},\$$

and so

$$\mathbf{N}'_{r-1}(t) \in \operatorname{span}\{\boldsymbol{\gamma}'(t), \dots, \boldsymbol{\gamma}^{(r)}(t), \boldsymbol{\gamma}^{(r+1)}(t)\} = \operatorname{span}\{\mathbf{T}(t), \mathbf{N}_1(t), \dots, \mathbf{N}_{r-1}(t)\}.$$

The Frenet-Serret formulas are

$$\frac{d\mathbf{T}}{ds} = \kappa_1 \mathbf{N}_1,$$

$$\frac{d\mathbf{N}_1}{ds} = -\kappa_1 \mathbf{T} + \kappa_2 \mathbf{N}_2,$$

$$\frac{d\mathbf{N}_i}{ds} = -\kappa_i \mathbf{N}_{i-1} + \kappa_{i+1} \mathbf{N}_{i+1}, \qquad 2 \le i \le r - 2,$$

$$\frac{d\mathbf{N}_{r-1}}{ds} = -\kappa_{r-1} \mathbf{N}_{r-2}.$$

The formulas for $\kappa_i(t)$, $1 \le i \le r - 1$, given in Theorem 3.6, still hold.

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