ON DISCRETE MORSE-BOTT THEORY

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ABSTRACT. This paper shows that discrete Morse–Bott theory can be developed in an intuitive way, which is achieved by improving the definition of the discrete Morse–Bott function originally introduced by S. Yaptieu. In fact, we demonstrate that various natural properties hold for discrete Morse–Bott functions and, in particular, establish the discrete Morse–Bott inequality, which can be regarded as an extension of both the discrete Morse inequalities and the continuous Morse–Bott inequalities.

1. Introduction

Morse theory has been developed as a framework to describe the topological structure of manifolds through the critical points of smooth functions (cf. [8]). It establishes a correspondence between the properties of critical points of Morse functions and the topology of manifolds. R. Bott [4] generalized Morse theory to Morse–Bott theory by allowing isolated critical points to be replaced by critical submanifolds. Morse–Bott theory is useful for studying functions with symmetries or conserved quantities and has applications in symplectic geometry (cf. [1, 3]).

R. Forman [5] introduced discrete Morse theory, which is a combinatorial version of Morse theory defined on CW complexes. Thus, it is suitable for computing homology and simplifying cell complexes, making it useful in various fields such as topological data analysis [6, 9].

S. Yaptieu [10] proposed a generalized version of discrete Morse theory, called a discrete Morse–Bott function, where isolated critical points are replaced by more general subsets. This generalization reflects the structure of Morse–Bott theory in a discrete setting and allows the study of discrete functions with symmetry. However, the definition in the preprint slightly differs from intuitive expectations, which implies the existence of some gaps (e.g. Remark 2.1(ii), the proofs of Lemmas 2.3, 2.4, 2.8). Therefore, we improve the definition to fill these gaps and make it more intuitive by extending the concept of critical points in discrete Morse theory to allow critical sets. In fact, we demonstrate a discrete Morse–Bott inequality, which naturally generalizes the discrete Morse inequalities known from Morse theory and can be derived from the continuous Morse–Bott theory. The inequality gives an upper bound for the Betti numbers of the underlying complex based on the structure of the critical sets. Furthermore, it opens the way for the analysis of data with group actions or symmetries.

The present paper consists of four sections. In the next section, we review the fundamental concepts and properties of discrete Morse theory. In § 3, we introduce

Date: November 12, 2025.

²⁰²⁰ Mathematics Subject Classification. Primary 37B30; Secondary 05C70.

Key words and phrases. Morse-Bott function, discrete Morse theory, CW complexes, Poincaré polynomial.

discrete Morse-Bott function, which is a generalization of discrete Morse function (Theorem 3.2). In the final section, we observe that various properties that hold for discrete Morse functions also hold for discrete Morse-Bott functions, and establish the discrete Morse-Bott inequality (Theorem 4.12), which can be regarded as an extension of the discrete Morse inequalities (Proposition 4.13).

2. Preliminaries

In this section, we review the fundamental concepts and properties of discrete Morse theory.

2.1. Fundamental concepts. We define CW complex as follows.

Definition 1. Let X, Y be topological spaces and $f: A \to X$ a continuous map on a subspace $A \subseteq Y$. Then we define

$$X \cup_f Y := (X \sqcup Y) / \sim,$$

where the equivalence relation \sim is the smallest one such that for every $a \in A$, we have $a \sim f(a)$. The map f is called an **attaching map**.

Definition 2. A CW complex X is constructed inductively as follows:

- (i) $X^{-1} = \emptyset$.
- (ii) For any integer $n \geq 0$, suppose the topological space X^{n-1} has been defined. Then we set

$$X^n := X^{n-1} \cup_{f^n} \coprod_{\lambda \in \Lambda^n} D^n_{\lambda}$$

where D^n is the *n*-dimensional closed disk and $f^n:\coprod_{\lambda\in\Lambda^n}\partial D^n_\lambda\to X^{n-1}$ is the attaching map.

We define $X := \bigcup_{n \in \mathbb{Z}_{>0}} X^n$ and equip X with the topology such that

 $U \subset X$ is open if, for any $n \in \mathbb{Z}_{>0}$, $U \cap X^n$ is open in X^n .

The map

$$\varphi:D^n\hookrightarrow\coprod_{\lambda\in\Lambda^n}D^n_\lambda\twoheadrightarrow X^n\hookrightarrow X$$

is called a **characteristic map** and $e^n := \varphi(\operatorname{Int} D^n)$ is called a **cell**, where \hookrightarrow denotes an injection and \twoheadrightarrow denotes a surjection. The integer n is the dimension of the cell, denoted $\dim e^n$.

By definition, note that we have $X = \coprod_{\lambda \in \Lambda} e_{\lambda}^n$.

For simplicity, we identify a CW complex \mathbb{K} with the family of all its cells. With a slight abuse of notation, we also use \mathbb{K} to denote this family. From now on, let \mathbb{K} be a CW complex. We have the following concepts and notations.

Definition 3. A cell σ of \mathbb{K} is a face of a cell τ of \mathbb{K} (or $\sigma < \tau$) if $\sigma \subseteq \overline{\tau} - \tau$.

Definition 4. A face σ of τ is a **facet** of τ (or $\sigma \prec \tau$) if dim $\sigma = \dim \tau - 1$.

Definition 5. A k-dimenonal face σ of τ is **regular** (or $\sigma^{\text{reg}} < \tau$) if the following conditions hold:

- (1) The restriction $\varphi_{\tau}|: \varphi_{\tau}^{-1}(\sigma) \to \sigma$ is a homeomorphism, where $D^{\dim \tau} \subset \mathbb{R}^{\dim \tau}$ is the closed unit disk and $\varphi_{\tau}: D^{\dim \tau} \to \mathbb{K}$ is the characteristic map of τ .
- (2) $\overline{\varphi_{\tau}^{-1}(\sigma)}$ is a closed k-ball.

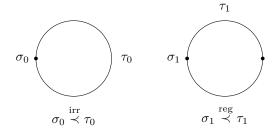
Definition 6. The relation $\sigma < \tau$ is denoted by $\sigma \prec \tau$ if $\dim \tau - \dim \sigma = 1$.

Definition 7. A cell σ is a **irregular** face of τ (or $\sigma < \tau$) if $\sigma < \tau$ but $\sigma \nleq \tau$.

Definition 8. The relation $\sigma \stackrel{\text{irr}}{<} \tau$ is denoted by $\sigma \stackrel{\text{irr}}{\prec} \tau$ if dim τ – dim $\sigma = 1$.

We provide the following examples to clarify the definitions.

Example 1. The following are examples of an irregular facet and a regular facet.



For a function on a CW complex, we call a cell nonritical when the value is not less than that on an adjacent higher-dimensional cell, as follows.

Definition 9. Let $f: \mathbb{K} \to \mathbb{R}$ be a function defined on a CW complex \mathbb{K} . A cell σ is a **noncritical** face of τ (or $\sigma \stackrel{\text{nc}}{\prec} \tau$) if $\sigma \prec \tau$ and $f(\sigma) \geq f(\tau)$.

We also introduce the following notations.

Definition 10. For any cells $\nu < \tau$ (resp. $\nu \prec \tau$), when $f(\nu) > f(\tau)$, we write $\nu < \tau$ (resp. $\nu \prec \tau$).

Note that the concept of "noncritical" is also used in the definition of a discrete Morse–Bott function below.

We define the following "interval".

Definition 11. For any cells $\sigma, \tau \in \mathbb{K}$ with $\sigma < \tau$, define the intervals $(\sigma, \tau) := \{\alpha \in \mathbb{K} \mid \sigma < \alpha < \tau\}, [\sigma, \tau] := \{\alpha \in \mathbb{K} \mid \sigma \leq \alpha \leq \tau\}, [\sigma, \tau) := \{\alpha \in \mathbb{K} \mid \sigma \leq \alpha < \tau\},$ and $(\sigma, \tau] := \{\alpha \in \mathbb{K} \mid \sigma < \alpha \leq \tau\}.$

2.2. Fundamental properties. We have the following property.

Lemma 2.1. For any cells $\nu, \sigma, \tau \in \mathbb{K}$ with $\nu \stackrel{\text{reg}}{\prec} \sigma \stackrel{\text{reg}}{\prec} \tau$, there is a cell $\widetilde{\sigma} \neq \sigma$ such that $\nu \prec \widetilde{\sigma} \prec \tau$.

The above lemma follows from the more general statement, Lemma 2.3 below, where a proof will be given.

Lemma 2.2. For any cells $\nu, \tau \in \mathbb{K}$ with $\nu \stackrel{\text{reg}}{<} \tau$ and $\dim \tau - \dim \nu \geq 2$, there is a cell $\sigma \in (\nu, \tau)$ (i.e. $\sigma \in \mathbb{K}$ such that $\nu < \sigma < \tau$).

Proof. Assume that there are no such cells. Then $\varphi_{\tau}^{-1}(\nu) = \varphi_{\tau}^{-1}(\partial \tau)$, which contradicts $\nu \stackrel{\text{reg}}{<} \tau$. Therefore, there is a cell $\sigma \in \mathbb{K}$ such that $\nu < \sigma < \tau$.

Lemma 2.3. For any cells $\nu, \alpha, \tau \in \mathbb{K}$ with $\nu < \alpha \stackrel{\text{reg}}{\prec} \tau$, there is a cell $\beta \in (\nu, \tau)$ with $\beta \nleq \alpha$.

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Proof. By $\varphi_{\tau}^{-1}(\nu) \subset \varphi_{\tau}^{-1}(\overline{\alpha}) = \overline{\varphi_{\tau}^{-1}(\alpha)}$, we have $\varphi_{\tau}^{-1}(\nu) \subset \partial \varphi_{\tau}^{-1}(\alpha)$. From definition of "regular", there is a cell $\beta \in \mathbb{K}$ such that $\tau > \beta > \nu$ and $\beta \nleq \alpha$.

Lemma 2.4. Let $\nu, \tau \in \mathbb{K}$ be cells with $\nu < \tau$. If there is exactly one cell $\alpha \in (\nu, \tau)$, then $\alpha \stackrel{\text{irr}}{<} \tau$.

Proof. Assume that $\alpha \stackrel{\text{reg}}{<} \tau$. If $\dim \tau - \dim \alpha \geq 2$, then Lemma 2.2 implies that there is a cell $\beta \in \mathbb{K}$ such that $\alpha < \beta < \tau$, which contradicts the uniqueness of α . Thus $\dim \tau - \dim \alpha = 1$. However, Lemma 2.3 implies the existence of another cells between ν and τ , which contradicts the uniqueness of α .

Remark 1. In the previous lemma, the statement $\nu \stackrel{\text{irr}}{\prec} \alpha$ need not hold. In fact, there is a CW complex satisfies $X = \nu \cup \widetilde{\nu} \cup \alpha \cup \tau$, $\nu \stackrel{\text{reg}}{\prec} \alpha$ as in Figure 1.

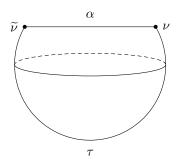


Figure 1. An example of a CW complex without an irregular cell of one higher dimension for any zero cell

2.3. **Discrete Morse function.** In this subsection, we review the Discrete Morse function, which will be extended in the next section.

Let $f: \mathbb{K} \to \mathbb{R}$ be a function defined on a CW complex \mathbb{K} . For any k-dimensional cell $\sigma^{(k)} \in \mathbb{K}$, set numbers $U(\sigma)$ and $D(\sigma)$ as follows:

$$U(\sigma) := \#\{\tau^{(k+1)} \mid \sigma \stackrel{\mathrm{nc}}{\prec} \tau\} = \#\{\tau^{(k+1)} \mid \sigma \prec \tau, f(\sigma) \geq f(\tau)\}$$

$$D(\sigma) := \#\{\nu^{(k-1)} \mid \nu \stackrel{\mathrm{nc}}{\prec} \sigma\} = \#\{\nu^{(k-1)} \mid \nu \prec \sigma, f(\nu) \geq f(\sigma)\}$$

Definition 12 (Discrete Morse function). The function f is **discrete Morse** if the following conditions holds for any $\sigma^{(k)} \in \mathbb{K}$:

- (M1) For any cell τ with $\sigma \stackrel{\text{irr}}{\prec} \tau$, we have $f(\sigma) < f(\tau)$.
- (M2) $U(\sigma) < 1$.
- (M3) For any cell ν with $\nu \stackrel{\text{irr}}{\prec} \sigma$, we have $f(\nu) < f(\sigma)$.
- (M4) $D(\sigma) \leq 1$.

By definition, we have the following observations.

Remark 2. From the definition (M1) of discrete Morse function, it follows that, for any discrete Morse function $f: \mathbb{K} \to \mathbb{R}$ and any cells $\sigma, \tau \in \mathbb{K}$, the condition $\sigma \stackrel{\text{nc}}{\prec} \tau$ implies $\sigma \stackrel{\text{reg}}{\prec} \tau$.

Lemma 2.5 (Lemma 6.11.[7]). For any discrete Morse function $f: \mathbb{K} \to \mathbb{R}$ and for any cell $\sigma \in \mathbb{K}$, we have the following inequality

$$U(\sigma) + D(\sigma) < 1$$

2.4. Fundamental concepts. We define fundamental concepts as follows.

Definition 13. Let f be a function defined on a CW complex. A cell σ is **critical** if $U(\sigma) = D(\sigma) = 0$.

Definition 14. A map $V: \mathbb{K} \to \mathbb{K} \sqcup \{0\}$ on a CW complex \mathbb{K} is a **combinatorial vector field** on \mathbb{K} if it satisfies the following conditions:

- (1) For any cell $\sigma \in V(\mathbb{K}) \setminus \{0\}$, we have $V(\sigma) = 0$.
- (2) For any cell $\sigma \in \mathbb{K} V^{-1}(0)$, we have $\sigma \stackrel{\text{reg}}{\prec} V(\sigma)$. (3) For any cell $\sigma \in \mathbb{K}$, we have $\#V^{-1}(\sigma) \leq 1$.

Definition 15. Let V be a combinatorial vector field on a CW complex \mathbb{K} . Define arrows \rightarrow as follows: $\sigma \rightarrow \tau$ if $V(\sigma) = \tau$.

Notice that $\sigma \to \tau$ implies $\sigma \stackrel{\text{reg}}{\prec} \tau$.

Definition 16. A path $\sigma_0 \to \tau_0 \succ \sigma_1 \to \tau_1 \succ \cdots \to \tau_{m-1} \succ \sigma_m$ is a **V-path** if for each $i, \sigma_i \neq \sigma_{i+1}$

Definition 17. A V-path $\sigma_0 \to \tau_0 \succ \sigma_1 \to \tau_1 \succ \dots \sigma_m \to \tau_m \succ \sigma_{m+1}$ is a closed **orbit** if $\sigma_{m+1} = \sigma_0$.

3. Discrete Morse-Bott function

In this section, we introduce a new definition of discrete Morse-Bott function by improving the original definition so that it also allows comparisons between cells that differ by at least two dimensions. With this improvement, results that one would naturally expect as analogues of the continuous case can be shown in a precise way.

Therefore, we modify the definition of a discrete Morse-Bott function as follows.

3.1. **Definition of discrete Morse-Bott function.** Let $f: \mathbb{K} \to \mathbb{R}$ be a function on a CW complex \mathbb{K} .

Definition 18. For any $r \in \mathbb{R}$, a sequence $(\sigma_0, \sigma_1, \dots, \sigma_l)$ is an **r-path** from a cell $\sigma_0 \in \mathbb{K}$ to a cell $\sigma_l \in \mathbb{K}$ if $\{\sigma_0, \sigma_1, \dots, \sigma_l\} \subseteq f^{-1}(r)$ and either $\sigma_i \prec \sigma_{i+1}$ or $\sigma_i \succ \sigma_{i+1}$ for any $i \in \{0, 1, \dots, l-1\}$.

Roughly speaking, an r-path is a path from a starting point to an endpoint that passes through cells of equal value whose dimensions differ by one, representing a kind of connectedness.

Definition 19 (Collection). A collection C for f is a maximal family of cells satisfying the following conditions:

- (1) There is a number $c \in \mathbb{R}$ such that $C \subset f^{-1}(c)$.
- (2) For any $\sigma, \tau \in C$, there is a c-path from σ to τ .

Denote by \mathcal{C}_f the set of collections of f. Then $\mathbb{K} = \bigsqcup \mathcal{C}_f$. For any cell $\sigma \in \mathbb{K}$, denote by $C_f(\sigma)$ the collection containing σ .

Notice that the condition (2) in the previous definition is used only in Remark 3, Theorem 3.2, and Proposition 4.13.

For any collection C for f and any k-dimensional cell $\sigma^{(k)} \in C$, set $U^C(\sigma)$ and $D^C(\sigma)$ as follows:

$$U^{C}(\sigma) := \#\{\tau^{(k+1)} \notin C \mid \sigma \stackrel{\mathrm{nc}}{\prec} \tau\} = \#\{\tau^{(k+1)} \notin C \mid \sigma \prec \tau, f(\sigma) \geq f(\tau)\}$$

$$D^C(\sigma) := \#\{\nu^{(k-1)} \not\in C \mid \nu \stackrel{\mathrm{nc}}{\prec} \sigma\} = \#\{\nu^{(k-1)} \not\in C \mid \nu \prec \sigma, f(\nu) \geq f(\sigma)\}$$

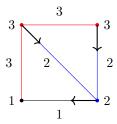
Using these notations, we introduce a discrete Morse-Bott function as follows.

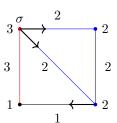
Definition 20 (Discrete Morse-Bott function). Let $\mathcal{C} := \{C^1, \dots, C^l\}$ be the family of collections for a function $f : \mathbb{K} \to \mathbb{R}$ on a CW complex \mathbb{K} . The function f is a **discrete Morse-Bott** function (with respect to \mathcal{C}) if the following conditions holds for any $i \in \{1, \dots, l\}$ and for all $\sigma^{(k)} \in C^i$:

- (MB1) For any cell τ with $\sigma \stackrel{\text{irr}}{<} \tau$, we have $f(\sigma) < f(\tau)$ (and so $\tau \notin C^i$).
- (MB2) $U^{C^i}(\sigma) \le 1$.
- (MB3) For any cell ν with $\nu \stackrel{\text{irr}}{<} \sigma$, we have $f(\nu) < f(\sigma)$ (and so $\nu \notin C^i$).
- (MB4) $D^{C^i}(\sigma) \leq 1$.

Note that we often use symbols ν , σ , and τ with the relation $\nu \prec \sigma \prec \tau$ among their dimensions, unless otherwise stated.

Example 2. The function on the left in the figure is a discrete Morse–Bott function. On the other hand, the function on the right does not, since it fails to satisfy (MB4) at a one-dimensional cell σ . Indeed, we obtain $U^{\mathcal{C}_f(\sigma)}(\sigma) = 2$, which contradicts (M4).





We have the following observation.

Lemma 3.1. Let $f: \mathbb{K} \to \mathbb{R}$ be a discrete Morse function. The function f is a discrete Morse-Bott function such that $\#C \le 2$ for any collection $C \in \mathcal{C}_f$.

Proof. Fix any $\sigma \in \mathbb{K}$. The condition (MB1) (resp. (MB3)) follows from the condition (B1) (resp. (B3)). The following conditions

$$U^{\mathcal{C}_f(\sigma)}(\sigma) \le U(\sigma) \le 1$$

$$D^{\mathcal{C}_f(\sigma)}(\sigma) \le D(\sigma) \le 1$$

imply the conditions (MB2) and (MB4). This means that f is a discrete Morse-Bott function.

Assume that $\#C \geq 3$. Then there is a c-path $\sigma_0, \sigma_1, \sigma_2$. Therefore, we obtain that $U(\sigma_1) + D(\sigma_1) \geq 2$, which contradicts $U(\sigma_1) + D(\sigma_1) = 1$ because of Lemma 2.5. Thus $\#C \leq 2$.

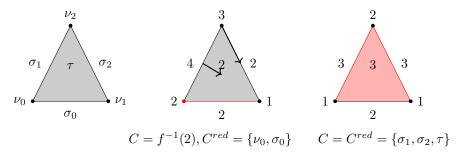
3.2. Reduced collection. Let $f: \mathbb{K} \to \mathbb{R}$ be a discrete Morse-Bott function. Recall that a cell $\sigma \in \mathbb{K}$ is **critical** if and only if $U(\sigma) = D(\sigma) = 0$. Fix any collection $C \in \mathcal{C}_f$. We introduce the following concepts.

Definition 21. A cell $\sigma \in C$ is upward noncritical if $U^C(\sigma) = 1$.

Definition 22. A cell $\sigma \in C$ is downward noncritical if $D^C(\sigma) = 1$.

Definition 23 (Reduced collection). The **reduced collection** C^{red} of C is defined as the set obtained from C by deleting all upward and downward noncritical cells in C.

Example 3. We describe below two examples of reduced collections.



Remark 3. The following are equivalent for any discrete Morse-Bott function $f : \mathbb{K} \to \mathbb{R}$ and any cell $\sigma \in \mathbb{K}$:

- (1) σ is critical.
- (2) $C = C^{\text{red}} = \{\sigma\}$, where $C \in \mathcal{C}_f$ is the collection containing σ .
- 3.3. Generalization of a discrete Morse function. We have the following equivalence, which demonstrates that the discrete Morse–Bott function is a generalization of the discrete Morse function.

Theorem 3.2. The following are equivalent for any function $f: \mathbb{K} \to \mathbb{R}$:

- (1) The function f is discrete Morse.
- (2) The function f is discrete Morse-Bott such that, for any collection $C \in \mathcal{C}_f$, either #C = 1 or both $C = C^{\text{red}}$ and #C = 2.

Proof. Suppose that f is discrete Morse. By Proposition 3.1, we have that $\#C \leq 2$ for any collection $C \in \mathcal{C}_f$. If #C = 2, then Lemma 2.5 implies that $0 = U^C(\sigma) + D^C(\sigma)$ and so that $C = C^{\text{red}}$, which implies assertion (2).

Conversely, suppose that assertion (2) holds. Fix any collection $C \in \mathcal{C}_f$ and any cell $\sigma \in C$. If C is a singleton (i.e. $C = \{\sigma\}$), then $U(\sigma) = U^C(\sigma)$ and $D(\sigma) = D^C(\sigma)$ and so (MB2) (resp. (MB4)) for C implies (B2) (resp. (B4)) for C. If #C = 2, then

$$U(\sigma) = \#\{\tau \mid \sigma \stackrel{\text{nc}}{\prec} \tau\} = \#\{\tau \notin C \mid \sigma \stackrel{\text{nc}}{\prec} \tau\} + \#\{\tau \in C \mid \sigma \stackrel{\text{nc}}{\prec} \tau\}$$

$$\leq 0 + 1 = 1$$

$$D(\sigma) = \#\{\tau \mid \tau \stackrel{\text{nc}}{\prec} \sigma\} = \#\{\tau \notin C \mid \tau \stackrel{\text{nc}}{\prec} \sigma\} + \#\{\tau \in C \mid \tau \stackrel{\text{nc}}{\prec} \sigma\}$$

$$\leq 0 + 1 = 1$$

and so (B2) and (B4) hold. This means that f is discrete Morse.

4. Natural properties of discrete Morse-Bott functions

In this section, we observe that various properties that hold for discrete Morse functions also hold for discrete Morse–Bott functions. To state them, we demonstrate some technical lemmas as follows.

Lemma 4.1. Let $f: \mathbb{K} \to \mathbb{R}$ be a discrete Morse-Bott function, $C \in \mathcal{C}_f$ a collection, and $\sigma \in C$ an upward noncritical cell. Then any cell $\nu \in C$ with $\nu \prec \sigma$ is upward noncritical.

Proof. Since $\sigma, \nu \in C$, we have $f(\sigma) = f(\nu)$. Since $\sigma \in C$ is upward noncritical, there is a cell $\tau \notin C$ with $\tau \stackrel{\text{nc}}{\succ} \sigma$. By definition of upward noncritical cell, we obtain $f(\sigma) > f(\tau)$. From $D^{\mathcal{C}_f(\tau)}(\tau) = 1$ and since $\sigma \notin \mathcal{C}_f(\tau)$ satisfies $\tau \stackrel{\text{nc}}{\succ} \sigma$, Lemma 2.1 implies there is a cell $\widetilde{\sigma}$ satisfying the following inequality:

$$f(\nu) = f(\sigma) > f(\tau) \ge f(\widetilde{\sigma})$$

From $\nu \prec \widetilde{\sigma}$, the cell ν is upward noncritical.

We have the following trichotomy.

Lemma 4.2. Let $f: \mathbb{K} \to \mathbb{R}$ be a discrete Morse-Bott function and $C \in \mathcal{C}_f$ a collection. Then one of the following statements holds exclusively for any cell $\sigma \in C - C^{\text{red}}$:

- (1) σ is upward noncritical.
- (2) σ is downward noncritical.

Proof. By definition of C^{red} , the cell σ satisfies either (1) or (2). Assume that the cell σ satisfies both (1) and (2). Then there are cells $\nu, \tau \notin C$ such that $\nu \stackrel{\mathrm{nc}}{\prec} \sigma \stackrel{\mathrm{nc}}{\prec} \tau$. Therefore, we have $\nu \stackrel{\mathrm{reg}}{\prec} \sigma \stackrel{\mathrm{reg}}{\prec} \tau$. By Lemma 2.1, there is a cell $\widetilde{\sigma} \neq \sigma \in \mathbb{K}$ such that $\nu \prec \widetilde{\sigma} \prec \tau$. This implies that

$$f(\tau) < f(\sigma) < f(\nu) \le f(\widetilde{\sigma}) \le f(\tau),$$

which is a contradiction.

Lemma 4.3. Let $f: \mathbb{K} \to \mathbb{R}$ be a discrete Morse-Bott function. For any cells $\nu, \tau \in \mathbb{K}$ with $\nu \stackrel{\text{snc}}{<} \tau$, there is $\sigma \in (\nu, \tau)$ with $\nu \stackrel{\text{snc}}{\prec} \sigma$.

Proof. Put $n := \dim \tau - \dim \nu$. We demonstrate the assertion for $\nu < \tau$ by induction on $n = \dim \tau - \dim \nu$. If n = 1, then one can choose $\sigma := \tau$. Thus, we may assume that $\dim \tau - \dim \nu > 1$ and the assertion for any pair $\nu' < \tau'$ with $\dim \tau' - \dim \nu' < n$ holds.

By Lemma 2.2, there is a cell $\alpha \in (\nu, \tau)$. Write $m := \dim \alpha - \dim \nu$. Suppose that m = 1. If $f(\alpha) \leq f(\tau)$, then $f(\alpha) \leq f(\tau) < f(\nu)$ and so one can choose $\sigma := \alpha$. Thus, we may assume that $f(\alpha) > f(\tau)$. By Lemma 2.3, there is a cell $\beta \in (\nu, \tau)$ with $\beta \nleq \alpha$. If $f(\beta) \leq f(\tau)$, then $f(\beta) \leq f(\tau) < f(\nu)$ and so the assertion holds by induction on n, because $\beta \in (\nu, \tau)$. On the other hand, assume that $f(\beta) > f(\tau)$. By the inductive hypothesis for $\beta < \tau$, there is a cell $\beta' \in (\beta, \tau)$ such that $\beta \prec \beta'$ and $f(\beta) > f(\beta')$. Since $\beta \nleq \alpha$, we have $\beta' \nleq \alpha$. Then $\nu < \beta' < \tau$ and $f(\beta) > f(\tau)$. By finite iterations of this process, one can obtain a cell $\beta^+ \in (\beta, \tau)$ with $\beta^+ \prec \tau$, $\beta^+ \nleq \alpha$, and $f(\beta^+) > f(\tau)$. Then $D^{C_f(\tau)}(\tau) > 1$, which contradicts $D^{C_f(\tau)}(\tau) \leq 1$.

Suppose that m > 1. If $f(\alpha) \le f(\tau)$, then $f(\nu) > f(\tau) \ge f(\alpha)$ and $\nu < \alpha$, which implies the assertion from the inductive hypothesis for $\nu < \alpha$. Thus, we may

assume that $f(\alpha) > f(\tau)$. Lemma 2.2 implies that there is a cell $\widetilde{\alpha} \in (\alpha, \tau) \subset (\nu, \tau)$. The inductive hypothesis for $\nu < \widetilde{\alpha}$ implies the assertion.

Lemma 4.4. Let $f: \mathbb{K} \to \mathbb{R}$ be a discrete Morse-Bott function and $C \in \mathcal{C}_f$. Then every cell $\sigma \in \overline{C^{\text{red}}} \setminus C^{\text{red}}$ with $f(\sigma) = f(C)$ is upward noncritical.

Proof. By $\sigma \in \overline{C^{\text{red}}}$, there is a cell $\tau \in C^{\text{red}}$ such that $\sigma < \tau$. Choose such a cell $\tau_0 \in C^{\text{red}}$ whose dimension is minimal. Since $f(\sigma) = f(C)$, we obtain $f(\sigma) = f(\tau_0)$.

Claim 1. If dim τ_0 – dim $\sigma = 1$, then σ is upward noncritical.

Proof. Suppose that $\dim \tau_0 - \dim \sigma = 1$. Assume that σ is downward noncritical. Then there is a cell $\nu \notin C$ with $\nu \stackrel{\text{snc}}{\prec} \sigma$. By Lemma 2.3, there is a cell $\widetilde{\sigma} \in (\nu, \tau_0)$ with $\widetilde{\sigma} \neq \sigma$. From $\tau_0 \in C^{\text{red}}$, we have $f(\widetilde{\sigma}) \leq f(\tau_0)$. By $\nu \notin C$ and $\tau_0 \in C$, we have $f(\nu) \leq f(\widetilde{\sigma}) \leq f(\tau_0) = f(\sigma) < f(\nu)$, which is a contradiction.

By the previous claim, we may assume that $\dim \tau_0 - \dim \sigma > 1$.

Claim 2. If there is no cell $\alpha \in (\sigma, \tau_0)$ such that $\alpha \stackrel{\text{reg}}{\prec} \tau_0$, then σ is upward noncritical.

Proof. Suppose that there is no cell $\alpha \in (\sigma, \tau_0)$ such that $\alpha \stackrel{\text{reg}}{\prec} \tau_0$. Moreover, assume that there is no cell $\beta \in (\sigma, \tau_0)$ such that $\beta \stackrel{\text{irr}}{<} \tau_0$. Then $\sigma \stackrel{\text{reg}}{\prec} \tau$. By the finiteness of dimension, Lemma 2.2 implies the following sequence:

$$\sigma \stackrel{\text{reg}}{\prec} \beta_1 \stackrel{\text{reg}}{\prec} \beta_2 \stackrel{\text{reg}}{\prec} \cdots \stackrel{\text{reg}}{\prec} \beta_k \stackrel{\text{reg}}{\prec} \tau_0$$

This contradicts the non-existence of α . Thus, there is a cell $\beta \in (\sigma, \tau_0)$ such that $\beta \stackrel{\text{irr}}{<} \tau_0$.

Then $f(\sigma) = f(\tau_0) > f(\beta)$. By $\sigma \stackrel{\text{snc}}{<} \beta$, Lemma 4.3 implies that there is a cell $\widetilde{\sigma} \in (\sigma, \beta)$ such that $\sigma \stackrel{\text{snc}}{<} \widetilde{\sigma}$. Therefore, σ is upward noncritical.

By the previous claim, we may assume that there is a cell $\alpha \in (\sigma, \tau_0)$ such that $\alpha \stackrel{\text{irr}}{\prec} \tau_0$.

By $\tau_0 \in C^{\text{red}}$, we obtain $f(\alpha) \leq f(\tau_0)$. Suppose that $f(\alpha) < f(\tau_0)$. Since $\sigma \stackrel{\text{snc}}{<} \alpha$, by $f(\alpha) < f(\tau_0) = f(\sigma)$, Lemma 4.3 implies that σ is upward noncritical. Therefore, we may assume that $f(\alpha) = f(\tau_0)$.

Claim 3. σ is upward noncritical.

Proof. Assume that σ is downward noncritical. Then there is a cell $\beta \prec \alpha$. By $\beta \prec \alpha \prec \tau_0$, Lemma 2.3 implies that there is a cell $\widetilde{\alpha} \neq \alpha$ with $\beta \prec \widetilde{\alpha} \prec \tau_0$. From $\tau_0 \in C^{\text{red}}$, we have the following contradiction:

$$f(\beta) > f(\alpha) = f(\tau_0) \ge f(\widetilde{\alpha}) \ge f(\beta)$$

By the previous claim, there is a cell $\alpha \stackrel{\text{nc}}{<} \gamma$. Since $f(\sigma) = f(\alpha) > f(\gamma)$, Lemma 4.3 implies that σ is upward noncritical.

Lemma 4.5. Let f be a discrete Morse–Bott function. For any collection $C \in \mathcal{C}_f$, the set difference $\overline{C^{\text{red}}} \setminus C^{\text{red}}$ is a subcomplex of \mathbb{K} .

Proof. Put $A := \overline{C^{\mathrm{red}}} \setminus C^{\mathrm{red}}$. Fix any cells $\nu < \sigma$ with $\sigma \in A$. It suffices to show that $\nu \in A$. By definition of closure, we have $\nu \in \overline{C^{\mathrm{red}}}$. If $\nu \notin C$, then $\nu \notin C^{\mathrm{red}}$ and so $\nu \in \overline{C^{\mathrm{red}}} - C^{\mathrm{red}} = A$. Thus, we may assume that $\nu \in C$. By $\sigma \in \overline{C^{\mathrm{red}}}$, there is a cell $\tau \in C^{\mathrm{red}}$ with $\sigma < \tau$. Choose such a cell $\tau \in C^{\mathrm{red}}$ whose dimension is minimal.

Suppose that there is no cell $\alpha \in (\nu, \tau_0)$ with $\alpha \prec \tau_0$. Assume that there is no cell $\beta \in (\nu, \tau_0)$ with $\beta \stackrel{\text{irr}}{<} \tau_0$. By $\sigma \in (\nu, \tau_0)$, we have that $\sigma \stackrel{\text{reg}}{<} \tau_0$. Applying Lemma 2.2 to $\sigma \stackrel{\text{reg}}{<} \tau_0$, there is a cell $\sigma' \in (\sigma, \tau_0)$ with $\sigma' \stackrel{\text{reg}}{<} \tau_0$. By finite iterations of this process, we have a cell $\alpha \in (\nu, \tau_0)$ with $\alpha \prec \tau_0$, which contradicts the non-existence of such a cell. Thus, there is a cell $\beta \in (\nu, \tau_0)$ with $\beta \stackrel{\text{irr}}{<} \tau_0$. Since $f(\beta) < f(\tau_0) = f(\nu)$, Lemma 4.3 implies that ν is upward noncritical. By definition of C^{red} , we obtain $\nu \in \overline{C^{\text{red}}} - C^{\text{red}} = A$.

Suppose that there is a cell $\alpha \in (\nu, \tau_0)$ with $\alpha \prec \tau_0$. By $\tau_0 \in C^{\text{red}}$, we have $f(\alpha) \leq f(\tau_0)$. If $f(\alpha) < f(\tau_0)$, then Lemma 4.3 implies that ν is upward noncritical and so $\nu \in \overline{C^{\text{red}}} - C^{\text{red}} = A$. Thus, we may assume that $f(\alpha) = f(\tau_0)$. By Lemma 4.4, the cell α is upward noncritical. Lemma 4.3 implies that ν is upward noncritical and so $\nu \in \overline{C^{\text{red}}} - C^{\text{red}} = A$.

For any cells $\tau, \sigma \in \mathbb{K}$ and any collection $C \in \mathcal{C}_f$, denote by $[\tau : \sigma]$ the incidence number. For any collection $C \in \mathcal{C}_f$, denote by $C_k(C^{\mathrm{red}}; \mathbb{Z})$ the free \mathbb{Z} -module generated by the k-cells of C^{red} . Put $C_{-1}(C^{\mathrm{red}}; \mathbb{Z}) := \{0\}$.

Definition 24. For any collection $C \in \mathcal{C}_f$ of a discrete Morse-Bott function and any integer $k \in \mathbb{Z}_{\geq 0}$, we define the boundary homomorphism

$$\partial_k^C: C_k(C^{\mathrm{red}}; \mathbb{Z}) \to C_{k-1}(C^{\mathrm{red}}; \mathbb{Z})$$

for any cell $\tau \in C^{\text{red}}$ by

$$\partial_k^C(\tau) := \sum_{\substack{\sigma \in C^{\mathrm{red}} \\ \sigma \prec \tau}} [\tau : \sigma] \sigma.$$

In the previous definition, the homomorphism is well-defined, independent of the orientation of the cells.

Proposition 4.6. For any collection $C \in \mathcal{C}_f$ of a discrete Morse-Bott function and any integer $k \in \mathbb{Z}_{>0}$, the equality

$$\partial_{k-1}^C \circ \partial_k^C = 0$$

holds.

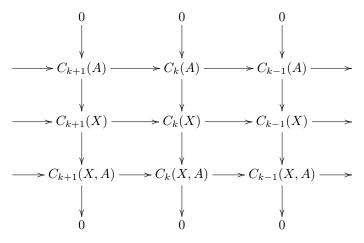
Proof. Put

$$X := \overline{C^{\mathrm{red}}}, \quad A := \overline{C^{\mathrm{red}}} \setminus C^{\mathrm{red}}.$$

Then both X and A are subcomplexes of \mathbb{K} . Since the restriction of the boundary homomorphism

$$\partial_k: C_k(\mathbb{K}) \to C_{k-1}(\mathbb{K})$$

to any subcomplex is also a boundary homomorphism, the following commutative diagram holds:



Here, the columns in the above diagram are short exact sequences, and the rows are chain complexes. Moreover, the boundary homomorphism ∂^C satisfies the following commutative diagram:

$$\longrightarrow C_{k+1}(X,A) \longrightarrow C_k(X,A) \longrightarrow C_{k-1}(X,A) \longrightarrow$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\longrightarrow C_{k+1}(C^{\text{red}}) \xrightarrow{\partial^C} C_k(C^{\text{red}}) \xrightarrow{\partial^C} C_{k-1}(C^{\text{red}}) \longrightarrow$$

Since $X \setminus A = C^{\text{red}}$, we have an isomorphism

$$C_k(C^{\mathrm{red}}) \cong C_k(X, A).$$

Therefore, it follows that

$$\partial^C \circ \partial^C = 0$$

Definition 25. For any collection $C \in \mathcal{C}_f$ of a discrete Morse-Bott function, define the k-th Betti number b_k^C of C and the Poincaré polynomial $P_t(C^{\text{red}})$ of C as follows:

$$b_k^C := \operatorname{rank}\left(Z_k^C/B_k^C\right),$$

$$P_t(C^{\mathrm{red}}) := \sum_k b_k^C t^k.$$

where $Z_k^C := \ker \partial_k^C$ and $B_k^C := \operatorname{im} \partial_{k+1}^C$.

Definition 26. For any discrete Morse function $f: \mathbb{K} \to \mathbb{R}$, define a combinatorial vector field $-\nabla f: \mathbb{K} \to \mathbb{K} \sqcup \{\emptyset\}$ as follows:

$$-\nabla f(\sigma) := \begin{cases} \tau & \text{if } \sigma \stackrel{\text{nc}}{\prec} \tau \\ \emptyset & \text{otherwise} \end{cases}$$

Definition 27. For any discrete Morse-Bott function $f: \mathbb{K} \to \mathbb{R}$, define a combinatorial vector field $-\nabla_s f: \mathbb{K} \to \mathbb{K} \sqcup \{\emptyset\}$ as follows:

$$-\nabla_s f(\sigma) := \begin{cases} \tau & \text{if } \sigma \stackrel{\text{snc}}{\prec} \tau \\ \emptyset & \text{otherwise} \end{cases}$$

Definition 28. Let V be a combinatorial vector field. We say that V is **positively bounded** if the following holds for any $\sigma \in \mathbb{K}$:

$$\sup\left(\{0\}\cup\left\{r\in\mathbb{Z}\ \middle|\ \text{there is a V-path containing no closed orbit}\right\}\right)<\infty.$$

We have the following characterization, which is essentially demonstrated in [7, p.95, Theorem 6.19].

Lemma 4.7. The following are equivalent for any positively bounded combinatorial vector field V:

- (1) There is a discrete Morse function f such that $V = -\nabla f$.
- (2) V has no closed orbits.

We omit the proof of the previous lemma, because it proceeds in exactly the same way as the original proof of [7, p.95, Theorem 6.19], except that the formula on line 19

$$f_p(\sigma^{p-1}) = f_{p-1}(\sigma) + \frac{d(\sigma)}{2D+1}$$

is replaced as follows:

$$f_p(\sigma^{p-1}) = f_{p-1}(\sigma) + \sum_{k=1}^{d(\sigma)} \frac{1}{2^k}$$

Lemma 4.8. Let f be a discrete Morse–Bott function. If $-\nabla_s f$ is positively bounded, then there is a discrete Morse function g with $-\nabla_s f = -\nabla g$.

Proof. Since $-\nabla_s f$ has no closed orbits, Lemma 4.7 implies the assertion.

We introduce the following notation to clarify the following discussion.

Definition 29. Let $f: \mathbb{K} \to \mathbb{R}$ be a function and L a subcomplex of \mathbb{K} . Then $c_k^L, m_k^L, u_k^L, d_k^L$ are defined as follows:

$$\begin{split} c_k^L &:= \#\{k\text{-cells of }L\} \\ m_k^L &:= \#\{\text{critical }k\text{-cells for }f \text{ in }L\} \\ u_k^L &:= \sum_{\sigma^k \in L} \#\{\tau \in L \mid \sigma \stackrel{\text{nc}}{\prec} \tau\} \\ d_k^L &:= \sum_{\sigma^k \in L} \#\{\nu \in L \mid \nu \stackrel{\text{nc}}{\prec} \sigma\} \end{split}$$

Then the following relations hold:

$$c_k^L = m_k^L + u_k^L + d_k^L$$
$$u_k^L = d_{k+1}^L$$

Moreover, we set

$$c_k := c_k^{\mathbb{K}}, \quad m_k := m_k^{\mathbb{K}}, \quad u_k := u_k^{\mathbb{K}}, \quad d_k := d_k^{\mathbb{K}}.$$

Proposition 4.9. Let f be a discrete Morse–Bott function on a finite CW complex \mathbb{K} . For any collection $C \in \mathcal{C}_f$, there is a polynomial r(t) with nonnegative integer coefficients such that

$$\sum_{k=0}^{\dim \mathbb{K}} \# \{ \sigma \in C^{\text{red}} \mid \dim \sigma = k \} t^k = P_t(C^{\text{red}}) + (1+t)r(t)$$

Moreover, the following equality holds:

$$r(t) = \sum_{k=1}^{\dim \mathbb{K}} \left(\operatorname{rank} B_{k-1}^{C}\right) t^{k-1}$$

Proof. For any integer $k \in \mathbb{Z}_{>0}$, the following two short exact sequences hold:

$$0 \to Z_k^C \to C_k(C^{\text{red}}) \to B_{k-1}^C \to 0$$

$$0 \to B_k^C \to Z_k^C \to Z_k^C/B_k^C \to 0$$

Then we obtain the following equality for any integer $k \in \mathbb{Z}_{>0}$:

$$\operatorname{rank} C_k(C^{\operatorname{red}}) = \operatorname{rank} Z_k^C + \operatorname{rank} B_{k-1}^C$$

$$\operatorname{rank} Z_k^C = \operatorname{rank} B_k^C + \operatorname{rank} (Z_k^C/B_k^C)$$

These imply the following equality:

$$\sum_{k=0}^{\dim \mathbb{K}} \#\{\sigma \in C^{\text{red}} \mid \dim \sigma = k\}t^k - P_t(C^{\text{red}}) = \sum_{k=0}^{\dim \mathbb{K}} \{\operatorname{rank} C_k(C^{\text{red}}) - b_k^C\}t^k$$

$$= \sum_{k=0}^{\dim \mathbb{K}} (\operatorname{rank} B_{k-1}^C + \operatorname{rank} B_k^C)t^k$$

$$= (1+t) \sum_{k=1}^{\dim \mathbb{K}} (\operatorname{rank} B_{k-1}^C)t^{k-1}.$$

By rank $B_{k-1}^C \geq 0$ for any integer $k \in \mathbb{Z}_{>0}$, any coefficient of r(t) is non-negative.

We state the following folklore result.

Theorem 4.10. Let f be a discrete Morse function on a finite CW complex \mathbb{K} . Then there is a polynomial r(t) with nonnegative integer coefficients such that

(1)
$$\sum_{k=0}^{\dim \mathbb{K}} m_k t^k = P_t(\mathbb{K}) + (1+t)r(t)$$

where $P_t(\mathbb{K}) = \sum_k b_k t^k$ is the Poincaré polynomisal of \mathbb{K} and

$$r(t) = \sum_{k=1}^{\dim \mathbb{K}} (\operatorname{rank} B_{k-1} - d_k).$$

Although this proof is similar to that of [2, Theorem 3.36] in the continuous setting, to the best of the authors' knowledge, no direct proof has been given in the literature, so we include it here for completeness.

Proof. For any $k \in \mathbb{Z}_{>0}$, from the two short exact sequences

$$0 \to Z_k \to C_k(\mathbb{K}) \to B_{k-1} \to 0$$

and

$$0 \to B_k \to Z_k \to Z_k/B_k \to 0$$
,

we obtain the following equality:

$$\operatorname{rank} C_k(\mathbb{K}) = \operatorname{rank} Z_k + \operatorname{rank} B_{k-1}$$

$$\operatorname{rank} Z_k = \operatorname{rank} B_k + \operatorname{rank} (Z_k/B_k) = \operatorname{rank} B_k + b_k$$

We have the following equality:

$$\sum_{k=0}^{\dim \mathbb{K}} m_k t^k - P_t(\mathbb{K}) = \sum_{k=0}^{\dim \mathbb{K}} \{\operatorname{rank} C_k(\mathbb{K}) - u_k - d_k - b_k\} t^k$$

$$= \sum_{k=0}^{\dim \mathbb{K}} (\operatorname{rank} B_{k-1} - d_k + \operatorname{rank} B_k - d_{k+1}) t^k$$

$$= (1+t) \sum_{k=1}^{\dim \mathbb{K}} (\operatorname{rank} B_{k-1} - d_k) t^{k-1}$$

In the sequence of equalities above, the relations $c_k = m_k + u_k + d_k$, $c_k = \operatorname{rank} C_k(\mathbb{K})$ for any $k \in \mathbb{Z}_{\geq 0}$, and the definition of $P_t(\mathbb{K})$ are applied in the first line. In the second line, the equality $b_k = \operatorname{rank}(Z_k/B_k)$ together with the two equalities above and the relation $d_{k+1} = u_k$ are used for any $k \in \mathbb{Z}_{\geq 0}$.

For a subcomplex L of \mathbb{K} , for any $k \in \mathbb{Z}_{\geq 0}$, denote by $\partial_{L,k} : C_k(L) \to C_{k-1}(L)$ the boundary homomorphism. For any $k \in \mathbb{Z}_{\geq 0}$, since rank $B_{k-1} - d_k = u_k + m_k - \text{rank } Z_k$, it suffices to show the inequality

$$(2) u_k + m_k - \operatorname{rank} Z_k \ge 0$$

for any subcomplex L. Assume that there is a subcomplex not satisfying the inequality (2). Fix a minimal such subcomplex K. Note that $L = \{\sigma^0\}$ satisfies the inequality (2). Let $\sigma \in K$ be one of the maximal cells of K such that

$$f(\sigma) = \max\{f(\alpha) \mid \alpha \text{ is a maximal cell in } K\}.$$

Then $K_{\sigma} := K \setminus \{\sigma\}$ is a subcomplex of \mathbb{K} . Put $p := \dim \sigma$. Since B is a minimal subcomplex which dose not satisfy (1), the sub complex $K \setminus \{\sigma\}$ satisfies (1). Since K_{σ} is obtained from K by removing the single cell σ , we obtain that rank ker $\partial_{K_{\sigma},p}$ – rank ker $\partial_{K_{\sigma},p} \in \{0,1\}$. Therefore, the inequality (1) implies the following equality:

$$(3) u_p^K + m_p^K = u_p^{K_\sigma} + m_p^{K_\sigma},$$

(4)
$$\operatorname{rank} \ker \partial_{K,p} = \operatorname{rank} \ker \partial_{K_{\sigma},p} + 1.$$

From (4), there is no maximal cell whose dimension is not k. Moreover, by (4), there is a k-cycle

$$x := \sum_{\alpha^k \in K} a_{\alpha} \alpha \in \ker \partial_{K,k}, \quad a_{\alpha} \in \mathbb{Z},$$

with $a_{\sigma} \neq 0$. By (3), the cell σ is downward noncritical, so there is $\nu \in K$ with $\nu \prec \sigma$. Since $\partial_{K,k} x = 0$, there is $\widetilde{\sigma} \in K \setminus \{\sigma\} = K_{\sigma}$ with $\widetilde{\sigma} \succ \nu$ such that $f(\widetilde{\sigma}) \geq f(\nu)$. Then $f(\widetilde{\sigma}) \geq f(\nu) > f(\sigma)$. Since dim $\widetilde{\sigma} = k$, the cell $\widetilde{\sigma}$ is maximal in K, which contradicts the choice of σ .

For any $k \in \mathbb{Z}_{>0}$, denote by \mathbb{K}^k the set of k-cells of \mathbb{K} .

Lemma 4.11. Let f be a discrete Morse–Bott function on a finite CW complex \mathbb{K} and $x \in \mathbb{Z}_k \setminus \{0\}$. Write

$$x = \sum_{\sigma \in \mathbb{K}^k} a_\sigma \, \sigma.$$

and define a non-empty set $A := \{ \sigma \in \mathbb{K}^k \mid a_{\sigma} \neq 0 \}$. For any cell $\alpha \in A$ with

(5)
$$f(\alpha) = \max\{f(\sigma) \mid \sigma \in A\}$$

and $\alpha \in C^{\text{red}}$ for some $C \in \mathcal{C}_f$, the sum

$$x^{\text{red}} := \sum_{\sigma \in A \cap C^{\text{red}}} a_{\sigma} \sigma$$

satisfies $x^{\text{red}} \in Z_k^C \setminus \{0\}.$

Proof. First, we show the following statement.

Claim 4. $\{\nu \prec \sigma \mid \nu \in C^{\text{red}}\} = \emptyset$ for any cell $\sigma \in A \setminus C^{\text{red}}$.

Proof. Fix any cell $\sigma \in A \setminus C^{\text{red}}$. Define $N_{\sigma} := \{ \nu \prec \sigma \mid \nu \in C^{\text{red}} \}$. Assume that $N_{\sigma} \neq \emptyset$. Fix any cell $\nu \in N_{\sigma}$. Since $\nu \in C^{\text{red}}$, the cell ν is not upward noncritical and so

$$f(\alpha) = f(\nu) \le f(\sigma)$$
.

Because $\sigma \in A$, by the maximality (5) of α , it follows $f(\nu) = f(\sigma)$ and so that $\sigma \in C \setminus C^{\text{red}}$.

If σ is upward noncritical, then by Lemma 4.1, ν is also upward noncritical, which contradicts that ν is not. Thus σ is downward noncritical. Then there is $\widetilde{\nu} \prec \sigma$ such that $f(\widetilde{\nu}) > f(\sigma)$. By $\partial_k x = 0$, there is $\widetilde{\sigma} \in A \setminus \{\sigma\}$ with $\widetilde{\nu} \prec \widetilde{\sigma}$. Then $f(\alpha) = f(\sigma) < f(\widetilde{\nu}) \le f(\widetilde{\sigma})$, which contradict the maximality (5) of α .

By the previous claim, we have the following relation:

$$\{\nu \in \mathbb{K} \mid \sigma \in A, \ \nu \prec \sigma\} = \{\nu \in \mathbb{K} \mid \sigma \in A, \ \nu \prec \sigma, \ \nu \notin C^{\text{red}}\}$$
$$\cup \{\nu \in \mathbb{K} \mid \sigma \in A \cap C^{\text{red}}, \ \nu \prec \sigma, \ \nu \in C^{\text{red}}\}.$$

Therefore, we have the following equality:

$$0 = \partial_k x = \sum_{\sigma \in A} \sum_{\nu \prec \sigma} a_{\sigma} [\sigma : \nu] \nu$$
$$= \sum_{\sigma \in A \cap C^{\text{red}}} \sum_{\substack{\nu \prec \sigma \\ \nu \in C^{\text{red}}}} a_{\sigma} [\sigma : \nu] \nu + \sum_{\sigma \in A} \sum_{\substack{\nu \prec \sigma \\ \nu \notin C^{\text{red}}}} a_{\sigma} [\sigma : \nu] \nu$$

By the independence of the basis of the chain complex, each term in the above equation vanishes. In particular, we have the following equality:

$$0 = \sum_{\sigma \in A \cap C^{\text{red}}} \sum_{\substack{\nu \prec \sigma \\ \nu \in C^{\text{red}}}} a_{\sigma}[\sigma : \nu] \nu = \partial_k^C x^{\text{red}}$$

We also introduce the following notation to clarify the following discussion.

Definition 30. Let $f: \mathbb{K} \to \mathbb{R}$ be a function and L a subcomplex of \mathbb{K} . Then $\tilde{m}_k^L, \tilde{u}_k^L, \tilde{d}_k^L$ are defined as follows:

$$\begin{split} \tilde{m}_k^L &:= \sum_{C \in \mathcal{C}_f} \# \{ \sigma^k \in L \mid \sigma \in C^{\mathrm{red}} \} \\ \tilde{u}_k^L &:= \sum_{\sigma^k \in L} \# \{ \tau \in L \mid \sigma \stackrel{\mathrm{snc}}{\prec} \tau \} \\ \tilde{d}_k^L &:= \sum_{\sigma^k \in L} \# \{ \nu \in L \mid \nu \stackrel{\mathrm{snc}}{\prec} \sigma \} \end{split}$$

Then the following relations hold:

$$c_k^L = \tilde{m}_k^L + \tilde{u}_k^L + \tilde{d}_k^L$$
$$\tilde{u}_k^L = \tilde{d}_{k+1}^L$$

Moreover, we set

$$\tilde{m}_k := \tilde{m}_k^{\mathbb{K}}, \quad \tilde{u}_k := \tilde{u}_k^{\mathbb{K}}, \quad \tilde{d}_k := \tilde{d}_k^{\mathbb{K}}.$$

Now, we show the following inequality.

Theorem 4.12. Let f be a discrete Morse–Bott function on a finite CW complex \mathbb{K} . Then there is a polynomial R(t) with nonnegative integer coefficients such that

(6)
$$\sum_{C \in \mathcal{C}_f} P_t(C^{\text{red}}) = P_t(\mathbb{K}) + (1+t)R(t)$$

where
$$R(t) := \sum_{k=1}^{\dim \mathbb{K}} \left(\operatorname{rank} B_{k-1} - \tilde{d}_k - \sum_{C \in \mathcal{C}_f} \operatorname{rank} B_{k-1}^C \right) t^{k-1}$$
.

Proof. Since \mathbb{K} is finite, the combinatorial vector field $-\nabla_s f$ is positively bounded. By Lemma 4.8, there is a discrete Morse function g such that $-\nabla_s f = -\nabla g$. Applying Theorem 4.10 to g, we obtain the following equality:

$$\sum_{k=0}^{\dim \mathbb{K}} m_k^g t^k = P_t(\mathbb{K}) + (1+t)r(t),$$

where

$$r(t) = \sum_{k=1}^{\dim \mathbb{K}} (\operatorname{rank} B_{k-1} - d_k^g) t^{k-1}.$$

The relation $-\nabla_s f = -\nabla g$ implies $m_k^g = \tilde{m}_k$. Therefore, the following holds:

$$\sum_{k=0}^{\dim \mathbb{K}} m_k^g t^k = P_t(\mathbb{K}) + (1+t)r(t)$$

$$\Rightarrow \sum_{k=0}^{\dim \mathbb{K}} \tilde{m}_k t^k = P_t(\mathbb{K}) + (1+t)r(t)$$

$$\Rightarrow \sum_{k=0}^{\dim \mathbb{K}} \sum_{C \in \mathcal{C}_f} \# \{ \sigma \in C^{\text{red}} \mid \dim \sigma = k \} t^k = P_t(\mathbb{K}) + (1+t)r(t)$$

$$\Rightarrow \sum_{C \in \mathcal{C}_f} (P_t(C^{\text{red}}) + (1+t) \sum_{k=1}^{\dim \mathbb{K}} (\operatorname{rank} B_{k-1}^C) t^{k-1}) = P_t(\mathbb{K}) + (1+t)r(t)$$

$$\Rightarrow \sum_{C \in \mathcal{C}_f} P_t(C^{\text{red}}) = P_t(\mathbb{K}) + (1+t)$$

$$\sum_{k=1}^{\dim \mathbb{K}} (\operatorname{rank} B_{k-1} - d_k^g - \sum_{C \in \mathcal{C}_f} \operatorname{rank} B_{k-1}^C) t^{k-1}$$

In the sequence of equalities above, the definition of \tilde{m}_k is applied in the third line, Proposition 4.9 in the fourth, and the definition of r(t) in the fifth.

It suffices to show that the coefficients of R(t) are nonnegative. Fix any $k \in \{1, 2, ..., \dim \mathbb{K}\}$. The following equalities

$$\operatorname{rank} C_k(\mathbb{K}) = \operatorname{rank} Z_k + \operatorname{rank} B_{k-1}$$

$$\operatorname{rank} C_k(C^{\operatorname{red}}) = \operatorname{rank} Z_k^C + \operatorname{rank} B_{k-1}^C$$

$$c_k^L = \tilde{m}_k^L + \tilde{u}_k^L + \tilde{d}_k^L$$

imply the following equality:

$$\operatorname{rank} B_{k-1} - \tilde{d}_k - \sum_{C \in \mathcal{C}_f} \operatorname{rank} B_{k-1}^C = \tilde{u}_k + \sum_{C \in \mathcal{C}_f} \operatorname{rank} Z_k^C - \operatorname{rank} Z_k$$

For any subcomplex L of \mathbb{K} , put $\mathcal{C}_L := \mathcal{C}_{f|_L}$. Then it suffices to show the following inequality:

(7)
$$\tilde{u}_k^L + \sum_{C \in \mathcal{C}_L} \operatorname{rank} Z_k^C - \operatorname{rank} \ker \partial_{L,k} \ge 0$$

Assume that there is a subcomplex of \mathbb{K} that does not satisfy (7). Note that $L := \{\sigma^0\}$ satisfies (7). Hence there is a minimal subcomplex of \mathbb{K} not satisfying (7), and denote it by K. Define

$$M := \{ \sigma \in K \mid \sigma \text{ is maximal in } K \}.$$

For any $\sigma \in M$, put $K_{\sigma} := K \setminus \{\sigma\}$. Then K_{σ} is a subcomplex of \mathbb{K} and satisfies (7). Therefore, by the same argument in the proof of Theorem 4.10, the following two relations hold:

(8)
$$\tilde{u}_k^K + \sum_{C \in \mathcal{C}_K} \operatorname{rank} Z_k^C = \tilde{u}_k^{K_{\sigma}} + \sum_{C \in \mathcal{C}_{K_{\sigma}}} \operatorname{rank} Z_k^C,$$

(9)
$$\operatorname{rank} \ker \partial_{K,k} = \operatorname{rank} \ker \partial_{K_{\sigma},k} + 1.$$

From (9) it follows that $\dim \sigma = k$. Therefore, all elements of M are k-cells, and conversely, every k-cell belongs to M. Choose a cell $\alpha \in M$ such that $f(\alpha) = \max\{f(\sigma) \mid \sigma \in M\}$. By (9), there is a chain $x \in \ker \partial_{K,k}$ which can be written as

$$x = \sum_{\sigma^k \in K} a_\sigma \sigma$$

with $a_{\alpha} \neq 0$. Assume that α is downward noncritical. Then there is a cell $\nu \in K$ with $\nu \stackrel{\text{snc}}{\prec} \alpha$. Since $\partial_{K,k} x = 0$, there is $\widetilde{\sigma} \succ \nu$ with $\widetilde{\sigma} \neq \sigma$ such that $f(\alpha) = f(\sigma) < f(\nu) \leq f(\widetilde{\sigma})$, which contradicts the maximality of α .

Thus, there is $C \in \mathcal{C}_K$ with $\alpha \in C^{\text{red}}$. If the set difference $C \setminus \{\sigma\}$ is a collection, then Lemma 4.11 implies rank $Z_k^C \neq \text{rank } Z_k^{C \setminus \{\sigma\}}$, which contradicts (8). Thus, the set difference $C \setminus \{\sigma\}$ is a finite disjoint union of collections $C_1 \sqcup \cdots \sqcup C_l$. By Lemma 4.11, we have rank $Z_k^C \neq \sum_{i=1}^l \text{rank } Z_k^{C_i}$, which contradicts (8).

We show that the equality (6) is a generalization of the equality (1) as follows.

Proposition 4.13. Let f be a discrete Morse function on a finite CW complex \mathbb{K} . Then the equality (6) is reduced into the equality (1).

Proof. Let $C \in \mathcal{C}_f$ be a collection with $C^{\text{red}} = \{\sigma, \tau\}$ with $\sigma^p \stackrel{\text{reg}}{\prec} \tau$. By definition of regularity of faces,

we obtain that $P_t(C^{\text{red}}) = 0$.

Suppose that f is Morse.

Claim 5.
$$\sum_{C \in \mathcal{C}_f} P_t(C^{\text{red}}) = \sum_{k=0}^{\dim \mathbb{K}} m_k t^k$$
.

Proof. By Theorem 3.2, for any collection $C \in \mathcal{C}$, we obtain that either $\#\mathcal{C} = 1$ or both $\#\mathcal{C} = 2$ and $C = C^{\text{red}}$. For any collection $C \in \mathcal{C}$ with $\#\mathcal{C} = 2$, we have that $P_t(C^{\text{red}}) = 0$. For any collection $C \in \mathcal{C}$ with $\#\mathcal{C} = 1$ and $C^{\text{red}} = \emptyset$, we obtain that $P_t(C^{\text{red}}) = 0$. By Remark 3, any collection $C \in \mathcal{C}$ consists of one critical cell if and only if $\#\mathcal{C} = \#\mathcal{C}^{\text{red}} = 1$. By definition of m_k , the desired equality holds.

Claim 6.
$$d_k = \tilde{d}_k + \sum_{C \in \mathcal{C}_f} \operatorname{rank} B_{k-1}^C$$
.

Proof. For any collection $C \in \mathcal{C}_f$ with #C = 1, the definition of B_{k-1}^C implies rank $B_{k-1}^C = 0$. Therefore, we have the following equalty:

$$\sum_{C \in \mathcal{C}_f} \operatorname{rank} B_{k-1}^C = \sum_{C \in \mathcal{C}_f, \#C = 2} \operatorname{rank} B_{k-1}^C$$

On the other hand, the following equality holds:

$$\begin{split} & \sum_{\sigma \in \mathbb{K}} \#\{\nu \in \mathbb{K} \mid f(\nu) = f(\sigma), \nu \prec \sigma\} \\ & = \sum_{C \in \mathcal{C}} \sum_{\sigma \in C} \#\{\nu \in \mathbb{K} \mid f(\nu) = f(\sigma), \nu \prec \sigma\} \\ & = \sum_{C \in \mathcal{C}} \sum_{\sigma \in C} \#\{\nu \in C \mid \nu \prec \sigma\} \\ & = \sum_{C \in \mathcal{C}} \sum_{\sigma \in C \text{red}} \#\{\nu \in C^{\text{red}} \mid \nu \prec \sigma\} \end{split}$$

In the sequence of equalities above, the fact that any collection is contained in the inverse image of a point is applied in the second line. Since rank $B_{k-1}^C=0$ for any collection $C\in\mathcal{C}$ with #C=1, the third equality holds.

It suffices to show that rank $B_{k-1}^C = \sum_{\sigma^k \in C^{\text{red}}} \# \{ \nu \in C^{\text{red}} \mid \nu \prec \sigma \}$ for any collection $C \in \mathcal{C}_f$ with #C = 2.

Fix any collection $C \in \mathcal{C}_f$ with #C = 2. If any of rank B_{k-1}^C and $\sum_{\sigma^k \in C^{\text{red}}} \#\{\nu \in C^{\text{red}} \mid \nu \prec \sigma\}$ is non-zero, then it must contains a k-cell and a (k-1)-cell in C. By #C = 2, both rank B_{k-1}^C and $\sum_{\sigma^k \in C^{\text{red}}} \#\{\nu \in C^{\text{red}} \mid \nu \prec \sigma\}$ are either zero or one. Suppose that $\sum_{\sigma^k \in C^{\text{red}}} \#\{\nu \in C^{\text{red}} \mid \nu \prec \sigma\}$ is one. Then there are cells

Suppose that $\sum_{\sigma^k \in C^{\text{red}}} \# \{ \nu \in C^{\text{red}} \mid \nu \prec \sigma \} \text{ is one. Then there are cells } \nu \prec \sigma \in C.$ By definition of regularity, we have $\nu \prec \sigma \in C$. Then $\partial^C \sigma = a_\nu \nu$ for some non-zero a_ν . This implies rank $B_{k-1}^C = 1$.

Conversely, suppose that rank $B_{k-1}^C = 1$. By definition of ∂^C , there are cells $\nu \prec \sigma^k \in C = C^{\mathrm{red}}$. This means that $\sum_{\sigma^k \in C^{\mathrm{red}}} \#\{\nu \in C^{\mathrm{red}} \mid \nu \prec \sigma\} = 1$.

The previous claims imply the assertion.

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