Sobolev spaces $W_2^{1/2}$: simultaneous improvement of functions by a homeomorphism of the circle

Vladimir Lebedev

Abstract. It is known that for every continuous real-valued function f on the circle $\mathbb{T}$ there exists a change of variable, i.e., a self-homeomorphism h of $\mathbb{T}$, such that the superposition $f \circ h$ is in the Sobolev space $W_2^{1/2}(\mathbb{T})$. In this paper we obtain certain results on simultaneous improvement of functions by a single homeomorphism. The main result is as follows: there does not exist a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in C^{1/2}(\mathbb{T})$. Here $C^{1/2}(\mathbb{T})$ is the space of all functions on $\mathbb{T}$ satisfying the Lipschitz condition of order $1/2$.

2020 *Mathematics Subject Classification*: Primary 42A16.

Key words and phrases: harmonic analysis, homeomorphisms of the circle, superposition operators, Sobolev spaces.

1. Introduction. Given an integrable function f on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ consider its Fourier expansion:

$$f(t) \sim \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{ikt}, \qquad t \in \mathbb{T}.$$

Recall that the Sobolev space $W_2^{1/2}(\mathbb{T})$ is the space of all (integrable) functions f satisfying

$$\|f\|_{W_2^{1/2}(\mathbb{T})} = \left(\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2|k|\right)^{1/2} < \infty. \tag{1}$$

In what follows, by $C(\mathbb{T})$ we denote the space of all continuous complex-valued functions on $\mathbb{T}$ (with the usual sup -norm). Given a modulus of continuity ω, i.e., a nondecreasing continuous function on $[0, +\infty)$ with $\omega(0) = 0$, by $C^\omega(\mathbb{T})$ we denote the space of all complex-valued functions f on $\mathbb{T}$ satisfying $\omega(f, \delta) = O(\omega(\delta))$, $\delta \to +0$, where

$$\omega(f, \delta) = \sup_{|t_1-t_2|\leq\delta} |f(t_1) - f(t_2)|, \qquad \delta \geq 0,$$

is the modulus of continuity of f. For $0 < \alpha \leq 1$ we just write C^α instead of C^{δ^α}.

It is known that certain properties of functions in $C(\mathbb{T})$ related to their Fourier transform can be improved by an appropriate change of variable, i.e., a self-homeomorphism of $\mathbb{T}$. The first result in this area is due to Bohr and Pál, who proved that for every real-valued function f in $C(\mathbb{T})$ there exists a self-homeomorphism h of $\mathbb{T}$ such that the superposition $f \circ h$ belongs to the space $U(\mathbb{T})$ of functions with uniformly convergent Fourier series. In addition, the proof yields a condition on the decay of the Fourier coefficients of $f \circ h$; namely, $\widehat{f \circ h} \in \bigcap_{p>1} l^p(\mathbb{Z})$. Subsequently, for certain function spaces, naturally arising in harmonic analysis, the question of whether every continuous function can be transformed by a suitable homeomorphic change of variable into a function that belongs to a given space, was studied by various authors. Some of these studies concern the possibility of simultaneous improvement of several functions by means of a single change of variable. For a survey on the subject see [3], [10]. More recent results are obtained in [2], [5–8]. [1]

The following improved version of the Bohr–Pál theorem was obtained in [11] (see also [5, Sec. 3], [7], [2]): for every real-valued $f \in C(\mathbb{T})$ there exists a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$. (Recall that, as is well known, $W_2^{1/2} \cap C(\mathbb{T}) \subseteq U(\mathbb{T})$.) It is worth noting that while the original proof of the Bohr–Pál theorem is based on the Riemann's theorem on conformal mappings, subsequent investigations mostly involve real-analytic methods.

The first result on simultaneous improvement of functions was obtained in [4] (see also [3, Sec. 2], [10, Sec. 4]). Namely: if K is a compact set in $C(\mathbb{T})$, then there exists a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in U(\mathbb{T})$ for every $f \in K$. Equivalently, this means that given an ω one can get $f \circ h \in U(\mathbb{T})$ for every $f \in C^\omega(\mathbb{T})$. This result naturally leads to a question whether it is possible to obtain $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in K$. The negative answer was obtained in [5, Th. 4], as it turned out, in general, there is no single change of variable which will bring two real-valued functions in $C(\mathbb{T})$ into $W_2^{1/2}(\mathbb{T})$. In other words there exists a complex-valued $f \in C(\mathbb{T})$ such that $f \circ h \notin W_2^{1/2}(\mathbb{T})$ whenever h is a self-homeomorphism of $\mathbb{T}$.

In this paper we obtain some further results on simultaneous improvement of functions in relation with the space $W_2^{1/2}(\mathbb{T})$.

In Sec. 2 we consider the family of all translations S_F of an arbitrary

[1]We note that in [5] $C(\mathbb{T})$ stands for the space of all *real-valued* continuous functions on $\mathbb{T}$.

function $F \in C(\mathbb{T})$:

$$S_F = \{F(\cdot + \theta),\, \theta \in \mathbb{T}\}$$

and show that if F is of analytic type (see the definition in Sec.2) and F is not in $W_2^{1/2}(\mathbb{T})$, then there is no homeomorphism which will bring all functions in S_F into $W_2^{1/2}(\mathbb{T})$.

In Sec. 3 for an arbitrary function $F \in C(\mathbb{T})$ we consider the family K_F of all convolutions of F with probability measures:

$$K_F = \{F * \lambda,\, \lambda \in P(\mathbb{T})\},$$

where $P(\mathbb{T})$ is the set of all probability measures on $\mathbb{T}$. We show that if $F \notin W_2^{1/2}(\mathbb{T})$, then there is no homeomorphism which will bring all functions in K_F into $W_2^{1/2}(\mathbb{T})$.

We note that both S_F and K_F are compact sets in $C(\mathbb{T})$. and obviously $S_F \subseteq K_F$. It is also obvious that if $\lambda \in P(\mathbb{T})$, then $|\widehat{F * \lambda}(k)| \leq |\widehat{F}(k)|$, $k \in \mathbb{Z}$. Thus, if $F \in W_2^{1/2}(\mathbb{T})$, then S_F and K_F are contained in $W_2^{1/2}(\mathbb{T})$ and there is nothing to improve.

In Sec. 4 we obtain the main result of the paper. Namely, we show that there is no homeomorphism which will bring all functions in $C^{1/2}(\mathbb{T})$ into $W_2^{1/2}(\mathbb{T})$. This result is the direct consequence of the result on translations (as well as of that on convolutions) and the known fact that $C^{1/2}(\mathbb{T}) \not\subseteq W_2^{1/2}(\mathbb{T})$. We note that earlier it was shown [6] that if $\alpha < 1/2$ then there exist two real-valued functions in $C^\alpha(\mathbb{T})$ such that there is no single change of variable which will bring them into $W_2^{1/2}(\mathbb{T})$. The author does not know if such a pair of functions can be found in $C^{1/2}(\mathbb{T})$. It is also worth noting that if $\alpha > 1/2$ then the functions in $C^\alpha(\mathbb{T})$ do not require an improvement since for these α we have $C^\alpha(\mathbb{T}) \subseteq W_2^{1/2}(\mathbb{T})$. The imbedding follows from the known equivalence of the seminorm $\|\cdot\|_{W_2^{1/2}(\mathbb{T})}$ (see (1)) and the seminorm

$$|||f|||_{W_2^{1/2}(\mathbb{T})} = \left(\int\int_{[0,2\pi]^2} \frac{|f(x) - f(y)|^2}{|x-y|^2} dx dy \right)^{1/2}. \qquad (2)$$

The concluding Sec. 5 contains certain remarks, open problems and the shortest, known to the author, proof of the refined version of the Bohr–Pál theorem.

2. Translations of a continuous function of analytic type. Let $F \in C(\mathbb{T})$. For each $\theta \in \mathbb{T}$ define the function F_θ by $F_\theta(t) = F(t+\theta)$. Consider the family S_F of translations of F:

$$S_F = \{F_\theta,\, \theta \in \mathbb{T}\}.$$

Clearly, S_F is a compact set in $C(\mathbb{T})$.

By $C^+(\mathbb{T})$ we denote the class of all continuous functions of analytic type on $\mathbb{T}$, i.e., of those $F \in C(\mathbb{T})$, which satisfy $\widehat{F}(k) = 0$ for all $k < 0$.

Clearly, if $F \in W_2^{1/2}(\mathbb{T})$, then $S_F \subseteq W_2^{1/2}(\mathbb{T})$. On the other hand the following theorem holds.

Theorem 1. *Let $F \in C^+(\mathbb{T})$. Suppose that $F \notin W_2^{1/2}(\mathbb{T})$. Then there does not exist a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in S_F$.*

To prove the theorem we need Lemma 1 below which has a technical character and will also be used in the next section. By $V(\mathbb{T})$ we denote the class of all functions of bounded variation on $\mathbb{T}$. Before we proceed to the lemma note that the bilinear form

$$B(x,y) = \frac{1}{2\pi}\int_{\mathbb{T}} x(t)dy(t)$$

is defined for $x \in C(\mathbb{T})$, $y \in V(\mathbb{T})$ and is invariant with respect to self-homeomorphisms of $\mathbb{T}$, namely, if h is a homeomorphism, then $B(x \circ h, y \circ h) = B(x,y)$. In addition we note that if $x, y \in W_2^{1/2}(\mathbb{T})$, then $\sum_{k\in\mathbb{Z}} |\widehat{x}(-k)\, ik\, \widehat{y}(k)| < \infty$.

Lemma 1. *Let $x \in C(\mathbb{T})$, $y \in V(\mathbb{T})$ and $x, y \in W_2^{1/2}(\mathbb{T})$. Then*

$$\text{(i)} \qquad \frac{1}{2\pi}\int_{\mathbb{T}} x(t)dy(t) = \sum_{k\in\mathbb{Z}} \widehat{x}(-k)\, ik\, \widehat{y}(k);$$

$$\text{(ii)} \qquad \left|\frac{1}{2\pi}\int_{\mathbb{T}} x(t)dy(t)\right| \le \|x\|_{W_2^{1/2}(\mathbb{T})}\|y\|_{W_2^{1/2}(\mathbb{T})}.$$

Proof. Part (ii) follows immediately from (i). To verify (i) note that since

$$\frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikt}dy(t) = -\frac{1}{2\pi}\int_{\mathbb{T}} y(t)de^{-ikt} = iky(k),$$

then (i) holds in the case when x is a trigonometric polynomial. In the general case it suffices to approximate x by the Fejér sums:

$$\sigma_N(x)(t) = \sum_{|k|\le N}\left(1-\frac{|k|}{N}\right)\widehat{x}(k)e^{ikt}.$$

Indeed, by what we have already proved, we see that

$$\frac{1}{2\pi}\int_{\mathbb{T}}\sigma_N(x)(t)dy(t) = \sum_{|k|\le N}\left(1-\frac{|k|}{N}\right)\widehat{x}(-k)\,ik\,\widehat{y}(k). \tag{3}$$

Let $N\to\infty$. Since the sequence of the polynomials $\sigma_N(x)$ converges uniformly to x, we have

$$\frac{1}{2\pi}\int_{\mathbb{T}}\sigma_N(x)(t)dy(t)\to\frac{1}{2\pi}\int_{\mathbb{T}}x(t)dy(t)$$

and it remains to note that the right-hand side in (3) tends to the right-hand side in (i). The lemma is proved.

Proof of Theorem 1. We have $F\in C(\mathbb{T})$ and

$$F(t)\sim\sum_{n\ge 0}c_n e^{int},$$

where, by the assumption,

$$\sum_{n\ge 0}|c_n|^2 n=\infty. \tag{4}$$

Suppose that, contrary to the assertion of the theorem, there exists a self-homeomorphism h of $\mathbb{T}$ such that $F_\theta\circ h\in W_2^{1/2}(\mathbb{T})$ for all $\theta\in\mathbb{T}$. Consider the sets $T_m\subseteq\mathbb{T}$, $m=1,2,\dots$, defined by

$$T_m=\{\theta\in\mathbb{T}:\|F_\theta\circ h\|_{W_2^{1/2}(\mathbb{T})}\le m\}.$$

Note that the sets T_m, $m=1,2,\dots$, are closed (we leave the proof to the reader). Since at the same time

$$\mathbb{T}=\bigcup_{m=1}^{\infty}T_m,$$

then, using the Baire category theorem, we see that there exists an m_0 and an interval $I \subseteq \mathbb{T}$ such that $\|F_\theta \circ h\|_{W_2^{1/2}(\mathbb{T})} \leq m_0$ for all $\theta \in I$. Replacing, if needed, h with $h + \gamma_I$, where γ_I is the center of I, one can assume that $I = (-\delta_0, \delta_0)$, where $0 < \delta_0 \leq \pi$. Thus,

$$\|F_\theta \circ h\|_{W_2^{1/2}(\mathbb{T})} \leq m_0 \quad \text{for all} \quad \theta \in (-\delta_0, \delta_0). \tag{5}$$

For $0 < \delta < \delta_0$ set

$$F^\delta(t) = \frac{1}{2\delta} \int_{-\delta}^{\delta} F(t+\theta) d\theta.$$

Note that for all δ, $0 < \delta < \delta_0$, we have $F^\delta \circ h \in W_2^{1/2}(\mathbb{T})$ and

$$\|F^\delta \circ h\|_{W_2^{1/2}(\mathbb{T})} \leq c \quad \text{for all} \quad \delta \in (0, \delta_0), \tag{6}$$

where $c > 0$ does not depend on δ. To see this it suffices to use (5) and the equivalence of the seminorms (1) and (2).

It is clear that F^δ is continuous and of bounded variation. So, $F^\delta \circ h$ is also continuous and of bounded variation. Using Lemma 1 (see (ii)) and (6), we see that for all $\delta \in (0, \delta_0)$

$$\left| \frac{1}{2\pi} \int_{\mathbb{T}} \overline{F^\delta(t)} dF^\delta(t) \right| = \left| \frac{1}{2\pi} \int_{\mathbb{T}} \overline{F^\delta \circ h(t)} d(F^\delta \circ h)(t) \right| \leq$$

$$\leq \|\overline{F^\delta \circ h}\|_{W_2^{1/2}(\mathbb{T})} \|F^\delta \circ h\|_{W_2^{1/2}(\mathbb{T})} \leq c^2, \tag{7}$$

where the bar stands for the complex conjugation.

Let $\chi_\delta = \frac{1}{2\delta} 1_{(-\delta,\delta)}$, where $1_{(-\delta,\delta)}$ is the indicator function of the interval $(-\delta, \delta)$. Obviously, $F^\delta = F * \chi_\delta$ and $\widehat{\chi_\delta}(k) = \frac{1}{2\pi} \frac{\sin k\delta}{k\delta}$ for $k \neq 0$. By Lemma 1 (see (i)) we obtain

$$\frac{1}{2\pi} \int_{\mathbb{T}} \overline{F^\delta(t)} dF^\delta(t) = \sum_{k \in \mathbb{Z}} \overline{\widehat{F^\delta}(k)} ik \widehat{F^\delta}(k) = i \sum_{k \in \mathbb{Z}} |\widehat{F^\delta}(k)|^2 k =$$

$$= i \sum_{k \in \mathbb{Z}} |\widehat{F}(k)|^2 |\widehat{\chi_\delta}(k)|^2 k = i \sum_{n \geq 1} |c_n|^2 \left(\frac{1}{2\pi} \frac{\sin n\delta}{n\delta} \right)^2 n.$$

Thus (see (7)), for all $\delta \in (0, \delta_0)$

$$\sum_{n \geq 1} |c_n|^2 \left(\frac{1}{2\pi} \frac{\sin n\delta}{n\delta} \right)^2 n \leq c^2.$$

Chose a positive integer N. We see that for all $\delta \in (0, \delta_0)$

$$\sum_{n=1}^{N} |c_n|^2 \left(\frac{\sin n\delta}{n\delta}\right)^2 n \leq (2\pi c)^2.$$

Let $\delta \to +0$. We obtain

$$\sum_{n=1}^{N} |c_n|^2 n \leq (2\pi c)^2,$$

which contradicts (4), since N was chosen arbitrarily. The theorem is proved.

3. Convolutions of a continuous function with probability measures. Let $P(\mathbb{T})$ be the class of all probability measures on $\mathbb{T}$. Given a function $F \in C(\mathbb{T})$ consider the family K_F of convolutions of F with the measures in $P(\mathbb{T})$:

$$K_F = \{F * \lambda,\, \lambda \in P(\mathbb{T})\}.$$

Clearly, K_F is a compact set in $C(\mathbb{T})$. As we mentioned in the introduction, if $F \in W_2^{1/2}(\mathbb{T})$, then $K_F \subseteq W_2^{1/2}(\mathbb{T})$. On the other hand the following theorem holds.

Theorem 2. *Let $F \in C(\mathbb{T})$. Suppose that $F \notin W_2^{1/2}(\mathbb{T})$. Then there does not exist a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in K_F$.*

As above, for a $\theta \in \mathbb{T}$ the function F_θ is defined by $F_\theta(t) = F(t + \theta)$. To prove Theorem 2 we will need two lemmas.

Lemma 2. *Let $x \in C(\mathbb{T})$. Let φ be a continuous mapping of $\mathbb{T}$ into itself. Then for every $\nu \in \mathbb{Z}$ the function $\widehat{x_\theta \circ \varphi}(\nu)$ is continuous on $\mathbb{T}$ and*

$$\frac{1}{2\pi} \int_0^{2\pi} |\widehat{x_\theta \circ \varphi}(\nu)|^2 d\theta = \sum_{k \in \mathbb{Z}} |\widehat{x}(k)|^2 |\widehat{e^{ik\varphi}}(\nu)|^2. \qquad (8)$$

Proof. The continuity of $\widehat{x_\theta \circ \varphi}(\nu)$ is obvious. To prove (8), assume first that x is a trigonometric polynomial. Then (there is a finite number of nonzero summands in the sum)

$$x_\theta(t) = x(t+\theta) = \sum_k \widehat{x}(k) e^{ik(t+\theta)},$$

whence

$$x_\theta \circ \varphi(t) = \sum_k \widehat{x}(k) e^{ik\varphi(t)} e^{ik\theta}.$$

So,

$$\widehat{x_\theta \circ \varphi}(\nu) = \sum_k \widehat{x}(k) \widehat{e^{ik\varphi}}(\nu) e^{ik\theta},$$

and (using Parseval's identity) we obtain (8). In the general case, given $x \in C(\mathbb{T})$ we obtain (8) by approximation of x by the Fejér sums. Indeed, by what we have already proved, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |(\sigma_N(x)_\theta \circ \varphi)^\wedge(\nu)|^2 d\theta = \sum_{|k| \leq N} |\widehat{x}(k)|^2 \left(1 - \frac{|k|}{N}\right)^2 |\widehat{e^{ik\varphi}}(\nu)|^2$$

and tending $N \to \infty$ we obtain (8). The lemma is proved.

It is clear that for the identity homeomorphism $h_0(t) = t$ we have $\|e^{ikh_0}\|_{W_2^{1/2}(\mathbb{T})} = |k|^{1/2}$, $k \in \mathbb{Z}$. The following lemma shows that the corresponding lower bound can not be improved by a change of variable.

Lemma 3. *Let h be a self-homeomorphism of $\mathbb{T}$. Let $k \in \mathbb{Z}$. Assume that $e^{ikh} \in W_2^{1/2}(\mathbb{T})$. Then $\|e^{ikh}\|_{W_2^{1/2}(\mathbb{T})} \geq |k|^{1/2}$.*

Proof. Applying Lemma 1 (see (ii)), we obtain

$$|k| = \left| \frac{1}{2\pi} \int_{\mathbb{T}} e^{ikx} de^{-ikx} \right| = \left| \frac{1}{2\pi} \int_{\mathbb{T}} e^{ikh(t)} de^{-ikh(t)} \right| \leq \|e^{ikh}\|^2_{W_2^{1/2}(\mathbb{T})}.$$

The lemma is proved.

Proof of Theorem 2. Assume, that there exists a homeomorphism h such that $(F * \lambda) \circ h \in W_2^{1/2}(\mathbb{T})$ for every measure $\lambda \in P(\mathbb{T})$. Let $M(\mathbb{T})$ be the Banach space of all measures μ on $\mathbb{T}$ with the usual norm $\|\mu\|_{M(\mathbb{T})}$, equal to the variation of μ. Each $\mu \in M(\mathbb{T})$ is a linear combination of two probability measures, so we see that $(F*\mu)\circ h \in W_2^{1/2}(\mathbb{T})$ for all $\mu \in M(\mathbb{T})$. Obviously the space $W_2^{1/2}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|\cdot\|^\circ_{W_2^{1/2}(\mathbb{T})} = \|\cdot\|_{L^2(\mathbb{T})} + \|\cdot\|_{W_2^{1/2}(\mathbb{T})}.$$

Consider the operator $Q : M(\mathbb{T}) \to W_2^{1/2}(\mathbb{T})$ defined by

$$Q\mu = (F*\mu)\circ h$$

Using standard argument, namely the closed graph theorem, we see that Q is a bounded operator (we leave to the reader to verify that all assumptions of the closed graph theorem hold). Thus for each $\mu \in M(\mathbb{T})$ we have

$$\|(F*\mu)\circ h\|_{W_2^{1/2}(\mathbb{T})} \le \|(F*\mu)\circ h\|^\circ_{W_2^{1/2}(\mathbb{T})} \le c\|\mu\|_{M(\mathbb{T})},$$

where $c > 0$ is independent of μ. In particular, for $\mu = \delta_\theta$, where δ_θ is the unit mass at θ, taking into account that $F_\theta = F*\delta_\theta$, we see that

$$\|(F_\theta \circ h\|_{W_2^{1/2}(\mathbb{T})} \le c \tag{9}$$

for all $\theta \in \mathbb{T}$.

Note that for each $k \in \mathbb{Z}$ we can find a positive integer $m(k)$ so that

$$\sum_{|\nu|\le m(k)} |\widehat{e^{ikh}}(\nu)|^2|\nu| \ge |k|/2.$$

Indeed, if $e^{ikh} \in W_2^{1/2}(\mathbb{T})$ the existence of $m(k)$ follows from Lemma 3, while if $e^{ikh} \notin W_2^{1/2}(\mathbb{T})$, the existence of $m(k)$ is obvious.

Chose now an arbitrary positive integer N and let $M(N) = \max_{|k|\le N} m(k)$. Then for all $k \in \mathbb{Z}$ with $|k| \le N$ we have

$$\sum_{|\nu|\le M(N)} |\widehat{e^{ikh}}(\nu)|^2|\nu| \ge |k|/2. \tag{10}$$

At the same time, applying Lemma 2, we see that

$$\frac{1}{2\pi}\int_0^{2\pi} |\widehat{F_\theta\circ h}(\nu)|^2 d\theta \ge \sum_{|k|\le N} |\widehat{F}(k)|^2|\widehat{e^{ikh}}(\nu)|^2,$$

whence (by multiplying by $|\nu|$ and summing up over $|\nu| \leq M(N)$) we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \sum_{|\nu|\leq M(N)} |\widehat{F_\theta \circ h}(\nu)|^2|\nu|d\theta \geq \sum_{|k|\leq N} |\widehat{F}(k)|^2 \sum_{|\nu|\leq M(N)} |\widehat{e^{ikh}}(\nu)|^2|\nu|.$$

Taking (9) and (10) into account, we see that

$$c^2 \geq \sum_{|k|\leq N} |\widehat{F}(k)|^2(|k|/2),$$

which contradicts the assumptions of the theorem since N was chosen arbitrarily. The theorem is proved.

4. The class $C^{1/2}(\mathbb{T})$. Main result. Obviously, if $\alpha > 1/2$ then $C^\alpha(\mathbb{T}) \subseteq W_2^{1/2}(\mathbb{T})$ (see (2)). On the other hand the function

$$F(t) = \sum_{n\geq 0} 2^{-n/2}e^{i2^n t},$$

is in $C^{1/2}(\mathbb{T})$ (see, e.g., [1, Ch. XI, Sec. 6]) but is not in $W_2^{1/2}(\mathbb{T})$. Thus, an immediate consequence of Theorem 1 as well as that of Theorem 3, is the following Theorem 3.

Theorem 3. *There does not exist a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in C^{1/2}(\mathbb{T})$.*

5. Remarks. 1. Given two real-valued functions u and v in $C^{1/2}(\mathbb{T})$, does there exists a change of variable h such that $u \circ h$ and $v \circ h$ are in $W_2^{1/2}(\mathbb{T})$? (This question was already mentioned in the introduction.)

2. If $F \in C(\mathbb{T})$ is real-valued and $\theta \in \mathbb{T}$, $\theta \neq 0$, does there exists h such that $F \circ h$ and $F_\theta \circ h$ are in $W_2^{1/2}(\mathbb{T})$? What if $F \in C^{1/2}(\mathbb{T})$?

3. It is unclear if one can replace $C^+(\mathbb{T})$ with $C(\mathbb{T})$ in Theorem 1.

4. For $s > 0$ the Sobolev space $W_2^s(\mathbb{T})$ is defined as the space of integrable functions f on $\mathbb{T}$ with $\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2|k|^{2s} < \infty$. It is known [5, Corollary 3] that if K is a compact set in $C(\mathbb{T})$, then there exists a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in \bigcap_{s<1/2} W_2^s(\mathbb{T})$ for every $f \in K$.

5. There exists a real-valued $f \in C(\mathbb{T})$ such that $f \circ h \notin \bigcup_{s>1/2} W_2^s(\mathbb{T})$ whenever h is a self-homeomorphism of $\mathbb{T}$. This is a simple consequence of

the inclusion $\bigcup_{s>1/2} W_2^s \cap C(\mathbb{T}) \subseteq A(\mathbb{T})$, where $A(\mathbb{T})$ is the Wiener algebra of absolutely convergent Fourier series, and the result obtained in [9] (see also [10, Th. 3.2]): there exists a real-valued $f \in C(\mathbb{T})$ such that $f \circ h \notin A(\mathbb{T})$, for every self-homeomorphism h.

6. We provide now the shortest, known to the author, proof of the refined version of the Bohr–Pál theorem. Implicitly it is contained in the proof of Theorem 4 in [5] and is based on the same idea as the original proof that involves the Riemann's theorem on conformal mappings. Suppose that $f \in C(\mathbb{T})$ is real-valued. Without loss of generality we may assume that $f(t) > 0$ for all $t \in \mathbb{T}$. Consider the curve γ in the complex plane $\mathbb{C}$ given by $\gamma(t) = f(t)e^{it}$, $t \in [0, 2\pi]$. This is a closed continuous curve without intersections. By Ω we denote the interior domain bounded by γ. Consider a conformal mapping G of the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ onto Ω. As is known, G extends to a homeomorphism of the closure $\overline{D}$ of D onto the closure $\overline{\Omega}$ of Ω and, being thus extended, G yields a homeomorphism of the circle $\partial D = \{z \in \mathbb{C} : |z| = 1\}$ onto the boundary $\partial\Omega$ of Ω. Retaining the notation G for the extension we see that the function $g(t) = G(e^{it})$ is of the form $g(t) = \gamma(h(t))$, where h is a self-homeomorphism of the segment $[0, 2\pi]$. At the same time, it is known that $\pi \sum_{n\geq 0} |\widehat{g}(n)|^2 n$ is the area of Ω. Thus, $\gamma \circ h \in W_2^{1/2}(\mathbb{T})$. It remains to observe that $f \circ h = |\gamma \circ h|$ and use the fact that for an arbitrary function F the condition $F \in W_2^{1/2}(\mathbb{T})$ implies $|F| \in W_2^{1/2}(\mathbb{T})$, which is obvious since the seminorms defined by (1) and (2) are equivalent.

A stronger result, based on the theorem on conformal mappings, is obtained in [2], see also [7].

References

1. N. K. Bary, *A treatise on trigonometric series*, Vols. I, II, Pergamon Press, Oxford 1964.

2. W. Jurkat, D. Waterman, *Conjugate functions and the Bohr–Pál theorem*, Complex Variables 12 (1989), 67–70.

3. J.-P. Kahane, *Quatre leçons sur les homéomorphismes du circle et les séries de Fourier*, in: Topics in Modern Harmonic Analysis, Vol. II, Ist. Naz. Alta Mat. Francesco Severi, Roma, 1983, 955–990.

4. J.-P. Kahane, Y. Katznelson, *Séries de Fourier des fonctions bornée*, Studies in Pure Math., in Memory of Paul Turán, Budapest, 1983, pp. 395–410. (Preprint, Orsay, 1978.)

5. V. V. Lebedev, *Change of variable and the rapidity of decrease of Fourier coefficients*, Matematicheskiĭ Sbornik, 181:8 (1990), 1099–1113 (in Russian). English transl.: Mathematics of the USSR-Sbornik, 70:2 (1991), 541–555. English transl. corrected by the author is available at: https://arxiv.org/abs/1508.06673

6. V. Lebedev, *The Bohr–Pál theorem and the Sobolev space* $W_2^{1/2}$, Studia Mathematica, 231:1 (2015), 73–81.

7. V. V. Lebedev, *A short and simple proof of the Jurkat–Waterman theorem on conjugate functions*, Functional Analysis and Its Applications, 51:2 (2017), 148–151.

8. Vladimir Lebedev, Alexander Olevskii, *Homeomorphic changes of variable and Fourier multipliers*, Journal of Mathematical Analysis and Applications, **481**:2 (2020) 123502, 1–11.

9. A. M. Olevskiĭ, *Change of variable and absolute convergence of Fourier series*, Soviet Math. Dokl., 23 (1981), 76–79.

10. A. M. Olevskiĭ, *Modifications of functions and Fourier series*, Russian Math. Surveys, 40 (1985), 181–224.

11. A. A. Saakjan, *Integral moduli of smoothness and the Fourier coefficients of the composition of functions*, Mathematics of the USSR-Sbornik, 38 (1981), 549–561.

School of Applied Mathematics, HSE University,
34 Tallinskaya St., Moscow, 123458, Russia
E-mail: lebedevhome@gmail.com