ISOPARAMETRIC HYPERSURFACES IN $\mathbb{S}^n \times \mathbb{R}^m$ AND $\mathbb{H}^n \times \mathbb{R}^m$

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ABSTRACT. We first show that every isoparametric hypersurface in $\mathbb{S}^n\times\mathbb{R}^m$ or $\mathbb{H}^n\times\mathbb{R}^m$ possesses a constant angle function with respect to the canonical product structure. Exploiting this rigidity, we achieve a complete classification of isoparametric and homogeneous hypersurfaces in these product spaces. Furthermore, in the product of any two real space forms, we prove that a hypersurface with both constant angle and constant principal curvatures must be isoparametric. Consequently, for hypersurfaces in $\mathbb{S}^n \times \mathbb{R}^m$ and $\mathbb{H}^n \times \mathbb{R}^m$, the conditions of having constant angle and constant principal curvatures are equivalent to being isoparametric.

1. INTRODUCTION

A smooth non-constant function $F : M \to \mathbb{R}$ on a Riemannian manifold M is called *transnormal* if there exists a smooth function $b : \mathbb{R} \to \mathbb{R}$ such that $\|\nabla F\|^2 = b(F)$, where ∇F denotes the gradient of F. If, in addition, there exists a continuous function $a : \mathbb{R} \to \mathbb{R}$ such that the Laplacian satisfies $\Delta F = a(F)$, then F is said to be *isoparametric* (cf. [42]). The regular level sets $\Sigma = F^{-1}(t)$ are correspondingly referred to as *transnormal* or *isoparametric hypersurfaces*, respectively. As observed by Élie Cartan, the transnormal condition implies that the level hypersurfaces are parallel, while the isoparametric condition further guarantees that these parallel hypersurfaces have constant mean curvatures. Moreover, in real space forms, Cartan proved that a hypersurface is isoparametric if and only if it has constant principal curvatures.

The classification of isoparametric hypersurfaces in the Euclidean space $\mathbb{R}^n$ and hyperbolic space $\mathbb{H}^n$ was completed by Cartan [3] and Segre [35] as early as in 1938. By contrast, the $\mathbb{S}^n$ case remained a subtle and long-standing problem—indeed, S. T. Yau listed it as Problem 34 in "Open Problems in Geometry" [34]. After decades of contributions from numerous mathematicians [1, 4–8, 15, 16, 22, 23, 25, 27–31, 36–38], a complete classification on the unit sphere $\mathbb{S}^n$ was finally achieved in 2020 [9]. A natural continuation of this classical theme is to study the classification of isoparametric hypersurfaces in the Riemannian product of two real space forms, $M^n_{c_1} \times M^m_{c_2}$ $(c_1, c_2 \in \{1, 0, -1\})$.

In order to classify isoparametric and homogeneous hypersurfaces in the product manifold $\mathbb{S}^2 \times \mathbb{S}^2$, Urbano [41] introduced in 2019 a natural product structure P on the tangent bundle of $\mathbb{S}^2 \times \mathbb{S}^2$, together with an associated angle function C defined on an oriented hypersurface Σ. These constructions, in fact, extend verbatim to any product of two real space forms $M^n_{c_1} \times M^m_{c_2}$ $(c_1, c_2 \in \{1, 0, -1\})$. Concretely, if a tangent vector

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decomposes as (v_1, v_2) according to the product splitting, the structure P is defined by:

$$\begin{aligned} P : \mathfrak{X}(M^n_{c_1} \times M^m_{c_2}) &\longrightarrow \mathfrak{X}(M^n_{c_1} \times M^m_{c_2}) \\ (v_1, v_2) &\longmapsto (v_1, -v_2). \end{aligned}$$

With respect to the product metric, this tensor field satisfies $P^2 = Id$, and is parallel. Let $\Sigma \subset M^n_{c_1} \times M^m_{c_2}$ be an orientable hypersurface with unit normal vector field N. With respect to the product metric, the associated *angle function* C is defined by

$$\begin{aligned} C : \Sigma &\longrightarrow [-1, 1] \\ x &\longmapsto \langle PN(x), N(x)\rangle, \end{aligned}$$

which measures the projection of the normal vector onto the ±1-eigenspaces of P. The extreme values $C = \pm1$ correspond to normals entirely contained in one factor, while $|C| < 1$ indicates a genuine tilt between the two factors.

Recent works establish the following rigidity for isoparametric hypersurfaces:

Theorem ([11, 17, 18, 41]) *In each of the spaces $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{R}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{R}^2$, $\mathbb{S}^n \times \mathbb{R}$, and $\mathbb{H}^n \times \mathbb{R}$ (with $n \geq 2$), all isoparametric hypersurfaces have constant angle.*

In this paper, we extend these results to higher dimension Euclidean factors:

Theorem 1.1. *Let Σ be a connected isoparametric hypersurface in $\mathbb{S}^n \times \mathbb{R}^m$ or $\mathbb{H}^n \times \mathbb{R}^m$. Then the associated angle function C is constant along Σ.*

Remark 1.2. *In a forthcoming paper, shall investigate the remaining product types $\mathbb{S}^n \times \mathbb{S}^m$, $\mathbb{S}^n \times \mathbb{H}^m$, and $\mathbb{H}^n \times \mathbb{H}^m$, and establish a corresponding constant–angle property for isoparametric hypersurfaces in these settings.*

Remark 1.3. *The case $n \geq 2$ in Theorem 1.1 is proved in Section 5. The argument used there does not apply when $n = 1$; nevertheless, Example 3.5 together with Theorem 1.4-(iii) yields a direct verification that Σ has constant angle function in the $n = 1$ case.*

Urbano [41] obtained a complete classification of isoparametric hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ by constructing an efficient global frame adapted to the natural complex structures on $\mathbb{S}^2$. Several subsequent works followed his strategy to treat other product models. In 2018 Julio–Batalla [24] classified isoparametric hypersurfaces with constant principal curvatures in $\mathbb{S}^2 \times \mathbb{R}^2$; later, dos Santos–dos Santos [13] treated the case $M^2_{c_1} \times M^2_{c_2}$ with $c_1 \neq c_2$. Gao–Ma–Yao [17] removed the constant principal curvatures assumption in [13] and completed the classification; in a related work [18] they developed refined geometric tools to treat $\mathbb{H}^2 \times \mathbb{H}^2$. All these approaches crucially exploit the fact that every two-dimensional real space form carries a natural complex structure, and therefore their arguments do not generalize to higher dimensions (for instance, among all spheres only $\mathbb{S}^2$ and $\mathbb{S}^6$ admit almost complex structures, while whether $\mathbb{S}^6$ carries a complex structure remains the well-known Hopf problem [39, 40]).

It is also noteworthy that Ge-Radeschi [19] obtained a foliated diffeomorphism classification of codimension one singular Riemannian foliations (e.g. isoparametric foliation) on all closed simply connected 4-manifolds (including $\mathbb{S}^2 \times \mathbb{S}^2$). In addition, Qian-Tang [32] provided an isoparametric hypersurface in $\mathbb{S}^n \times \mathbb{S}^n$ and computed its curvature properties as well as the spectrum of the Laplace–Beltrami operator. More recently, Cui [10] provided further examples of isoparametric hypersurfaces by restricting certain isoparametric functions on $\mathbb{S}^{2n+1}$ to the product $\mathbb{S}^n \times \mathbb{S}^n$.

From another perspective, building upon the local classification of constant angle hypersurfaces in [12], de Lima and Pipoli [11] obtained a complete classification of isoparametric hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. They proved the following result:

Theorem ([11]) *Isoparametric hypersurfaces in $M_c^n \times \mathbb{R}$ $(c = \pm 1)$ are precisely one of the following:*

(i) *horizontal slice $M_c^n \times \{t_0\}$;*
(ii) *a vertical cylinder over a complete isoparametric hypersurface in M_c^n;*
(iii) *a parabolic bowl in $\mathbb{H}^n \times \mathbb{R}$.*

The classification above is based on the concept of (M_s, ϕ)-graphs. However, a direct extension of this construction to vector-valued functions produces submanifolds of higher codimension rather than hypersurfaces, thus does not apply when the Euclidean factor has dimension $m > 1$.

We adopt a different approach. Inspired by Miyaoka [26] and through a focused analysis along the special principal direction $V = PN - CN$, we establish the following classification of isoparametric hypersurfaces in $M_c^n \times \mathbb{R}^m$ $(c = \pm 1)$. (Note that when $n = 1$, only the case $\mathbb{S}^1 \times \mathbb{R}^m$ needs to be considered, as $\mathbb{H}^1$ does not exist.)

Theorem 1.4. *Let Σ be a connected complete isoparametric hypersurface in $M_c^n \times \mathbb{R}^m$ $(c = \pm 1,\ m \geq 2)$, i.e., in $\mathbb{S}^n \times \mathbb{R}^m$ or $\mathbb{H}^n \times \mathbb{R}^m$. Up to ambient isometry, Σ is one of the following:*

(i) *$K_1 \times \mathbb{R}^m$, where K_1 is an isoparametric hypersurface in M_c^n. For $n = 1$ this reduces to $\{p\} \times \mathbb{R}^m$, $p \in \mathbb{S}^1$;*
(ii) *$M_c^n \times K_2$, where K_2 is an isoparametric hypersurface in $\mathbb{R}^m$;*
(iii) *$\Phi(\mathbb{R}^m) \subset \mathbb{S}^1 \times \mathbb{R}^m$, where $\Phi\colon \mathbb{R}^m \to \mathbb{S}^1 \times \mathbb{R}^m$ is the immersion defined by*
$$x \mapsto \Big(\cos\langle x, x_0\rangle,\ \sin\langle x, x_0\rangle,\ x\Big),$$
with $\langle \cdot, \cdot \rangle$ denoting the standard inner product on $\mathbb{R}^m$ and $x_0 \in \mathbb{R}^m \setminus \{0\}$ fixed;
(iv) *$\Psi(\mathbb{R}^{n+m-1}) \subset \mathbb{H}^n \times \mathbb{R}^m$, where $\Psi\colon \mathbb{R}^{n+m-1} \to \mathbb{H}^n \times \mathbb{R}^m$ is given by*
$$(t, x, y) \mapsto \Big(p(t, x),\ q(t, y)\Big),$$
with
$$p(t, x) = \cosh\Big(t\sqrt{\varepsilon}\Big)\gamma_1(x) + \sinh\Big(t\sqrt{\varepsilon}\Big)N_{\gamma_1}(x),$$
$$q(t, y) = \gamma_2(y) + t\sqrt{1-\varepsilon}N_{\gamma_2},$$
where $\gamma_1(x)$ is a horosphere in $\mathbb{H}^n$ with unit normal N_{γ_1}, $\gamma_2(y)$ is an affine hyperplane in $\mathbb{R}^m$ with constant unit normal N_{γ_2}, and $\varepsilon \in (0, 1)$ is a constant.

Remark 1.5. *In a forthcoming paper, we shall generalize this classification to the remaining product types $\mathbb{S}^n \times \mathbb{S}^m$, $\mathbb{S}^n \times \mathbb{H}^m$ and $\mathbb{H}^n \times \mathbb{H}^m$.*

As mentioned earlier, in real space forms, isoparametric hypersurfaces coincide with hypersurfaces having constant principal curvatures. However, these two notions are no longer equivalent in general Riemannian manifolds. For example, Rodríguez-Vázquez

[33] constructed non-isoparametric hypersurfaces with constant principal curvatures in the torus $\mathbb{T}^n$ ($n \geq 3$), while Ge-Tang-Yan [21] exhibited isoparametric hypersurfaces in $\mathbb{C}P^n$ whose principal curvatures are not constant.

As the second main result of this paper, we consider hypersurfaces in the product manifold $M^n_{c_1} \times M^m_{c_2}$, and establish the following theorem.

Theorem 1.6. *Let Σ be a connected hypersurface in $M^n_{c_1} \times M^m_{c_2}$. If Σ has constant angle and constant principal curvatures, then it is isoparametric.*

Combining Theorems 1.1, 1.4, with 1.6, we immediately obtain the following characterization.

Corollary 1.7. *Let Σ be a connected complete hypersurface in $\mathbb{S}^n \times \mathbb{R}^m$ or $\mathbb{H}^n \times \mathbb{R}^m$. Then Σ is isoparametric if and only if it has constant angle and constant principal curvatures.*

Furthermore, by combining Corollary 1.7 with Theorem 1.4, we obtain a classification of homogeneous hypersurfaces in $\mathbb{S}^n \times \mathbb{R}^m$ and $\mathbb{H}^n \times \mathbb{R}^m$. This result generalizes that of [11], which corresponds to the case $m = 1$.

Corollary 1.8. *Let Σ be a homogeneous hypersurface in $M^n_c \times \mathbb{R}^m$ ($c = \pm 1$, $m \geq 2$), i.e., in $\mathbb{S}^n \times \mathbb{R}^m$ or $\mathbb{H}^n \times \mathbb{R}^m$. Up to ambient isometries, Σ is one of the following:*

(i) *$K_1 \times \mathbb{R}^m$, where K_1 is a homogeneous hypersurface in M^n_c. In the case $n = 1$, this reduces to $\{p\} \times \mathbb{R}^m$ with $p \in \mathbb{S}^1$;*
(ii) *$M^n_c \times K_2$, where K_2 is a homogeneous hypersurface in $\mathbb{R}^m$;*
(iii) *The hypersurface described in Theorem 1.4-(iii);*
(iv) *The hypersurface described in Theorem 1.4-(iv).*

The paper is organized as follows. In Section 3, we prove Theorem 1.4 and verify the homogeneity of the hypersurfaces listed therein. Section 4 is devoted to the proof of Theorem 1.6. Finally, due to its length and technical nature, the proof of Theorem 1.1 is presented separately in Section 5.

2. Preliminaries

Let Σ be an orientable hypersurface in the product manifold $M^n_{c_1} \times M^m_{c_2}$ with global unit normal vector field N. For any vector field $X \in \mathfrak{X}(M^n_{c_1} \times M^m_{c_2})$, we denote by X^h its *horizontal component* tangent to $M^n_{c_1}$ and by X^v its *vertical component* tangent to $M^m_{c_2}$. Let A be the shape operator of Σ associated with N, and H the mean curvature of Σ. The natural projection maps are given by

$$\begin{aligned} \pi_1 : M^n_{c_1} \times M^m_{c_2} &\longrightarrow M^n_{c_1}, \\ (x, y) &\longmapsto x, \end{aligned} \qquad\qquad \begin{aligned} \pi_2 : M^n_{c_1} \times M^m_{c_2} &\longrightarrow M^m_{c_2}, \\ (x, y) &\longmapsto y. \end{aligned}$$

For each $(x, y) \in \Sigma$, we define

$$\Sigma_y = \pi_1\Big(\pi_2^{-1}(y) \cap \Sigma\Big) \quad \text{and} \quad \Sigma_x = \pi_2\Big(\pi_1^{-1}(x) \cap \Sigma\Big),$$

which represent the projections of Σ into the horizontal and vertical factors, respectively.

Decompose the unit normal vector as $N = (N^h, N^v)$. The angle function C is then accordingly given by

$$C = \langle PN, N \rangle = \|N^h\|^2 - \|N^v\|^2 = C_1^2 - C_2^2, \tag{2.1}$$

where

$$C_1 = \|N^h\| = \sqrt{\frac{1+C}{2}}, \quad C_2 = \|N^v\| = \sqrt{\frac{1-C}{2}}.$$

Now we introduce a special tangent vector field V on Σ, which will play an important role in the subsequent verification. It is defined by

$$V = PN - CN = \Big((1-C)N^h, -(1+C)N^v\Big). \tag{2.2}$$

It follows immediately that $\|V\|^2 = 1 - C^2$. Differentiating (2.1) and using the fact that P is parallel, we obtain, for any tangent vector field X on Σ,

$$\begin{aligned} X(C) &= \langle \nabla_X(PN), N \rangle + \langle PN, \nabla_X N \rangle \\ &= -2\langle AX, V \rangle = -2\langle X, AV \rangle. \end{aligned}$$

Hence, the gradient of C is given by

$$\nabla^\Sigma C = -2AV. \tag{2.3}$$

In the product $M_c^n \times \mathbb{R}^m (c = \pm 1)$, the Riemannian curvature tensor R_c of the product manifold $M_c^n \times \mathbb{R}^m$ is given by

$$R_c(X,Y)Z = c\Big(\langle X^h, Z^h \rangle Y^h - \langle Y^h, Z^h \rangle X^h\Big), \qquad \forall\, X, Y, Z \in \mathfrak{X}(M_c^n \times \mathbb{R}^m). \tag{2.4}$$

3. Classification of Isoparametric Hypersurfaces

In this section, we aim to prove Theorem 1.4. For the fluency of expression, we begin by preparing two propositions to characterize the focal points and principal frames of transnormal hypersurfaces with constant angle in general Riemannian product manifolds $M_1 \times M_2$.

Proposition 3.1. *Let Σ be a connected complete transnormal hypersurface in the Riemannian product $M_1 \times M_2$. If the angle function C is constant with $-1 < C < 1$, then for any $(x_0, y_0) \in \Sigma$, the slices Σ_{x_0} and Σ_{y_0} are transnormal hypersurfaces in M_2 and M_1, respectively.*

Moreover, if $(x, y) \in M_1 \times M_2$ is a focal point of Σ, then x is a focal point in M_1 and y is a focal point in M_2. Conversely, if x is a focal point in M_1 or y is a focal point in M_2, then (x, y) is a focal point in $M_1 \times M_2$.

Proof. Without loss of generality, let $\Sigma = F^{-1}(t)$ be a regular level set of a transnormal function $F : M_1 \times M_2 \to \mathbb{R}$ satisfying $\|\nabla F\|^2 = b(F)$. Denote by ∇^h and ∇^v the gradients on M_1 and M_2, respectively.

For fixed points $x_0 \in M_1$ and $y_0 \in M_2$, define

$$\begin{aligned} F_{x_0} : M_2 &\longrightarrow \mathbb{R}, & F_{y_0} : M_1 &\longrightarrow \mathbb{R}, \\ y &\longmapsto F(x_0, y), & x &\longmapsto F(x, y_0). \end{aligned}$$

Then $\Sigma_{x_0} = F_{x_0}^{-1}(t)$ and $\Sigma_{y_0} = F_{y_0}^{-1}(t)$. A straightforward computation yields

$$\|\nabla^v F_{x_0}(y)\|^2 = \|\nabla^v F(x_0, y)\|^2$$

$$= \frac{1-C}{2}\|\nabla F(x_0,y)\|^2 = \frac{1-C}{2} b\Big(F(x_0,y)\Big),$$

and the corresponding relation for F_{y_0} is analogous. Moreover, by (2.2)

$$\begin{aligned}\exp_{(x_0,y_0)} \frac{2}{1-C} t(0,N^v) &= \exp_{(x_0,y_0)} \frac{1}{1-C} t\Big((1-C)N - V\Big)\\ &= \exp_{\exp_{(x_0,y_0)}\left(-\frac{1}{1-C}tV\right)} tN,\end{aligned}$$

and similarly,

$$\begin{aligned}\exp_{(x_0,y_0)} \frac{2}{1+C} t(N^h,0) &= \exp_{(x_0,y_0)} \frac{1}{1+C} t\Big((1+C)N + V\Big)\\ &= \exp_{\exp_{(x_0,y_0)} \frac{1}{1+C}tV} tN.\end{aligned}$$

Since $-1 < C < 1$, we have $\|V\|^2 = 1 - C^2 \neq 0$. Hence V is a nonvanishing tangent vector field on the complete hypersurface Σ. Therefore, $\exp_{(x_0,y_0)} tV$ defines a diffeomorphism for each t, and the differential of $\exp_{y_0} tN^v$ (resp., $\exp_{x_0} tN^h$) has the same rank as that of $\exp_{(x_0,y_0)} tN$. The desired conclusion follows.

$\square$

Remark 3.2. *When $C \equiv 1$, we have $N = (N^h, 0)$. Thus, for any $(x_0,y_0) \in \Sigma$, $\Sigma_{y_0} = M_1$ and Σ_{x_0} is a transnormal hypersurface in M_2. Consequently, $(x,y) \in M_1 \times M_2$ is a focal point if and only if $y \in M_2$ is a focal point. The case $C \equiv -1$ is analogous.*

Proposition 3.3. *Let Σ be a connected complete transnormal hypersurface with constant angle function C in a Riemannian product $M_1^n \times M_2^m$, and set $V = PN - CN$. If $-1 < C < 1$, then at any point $(x,y) \in \Sigma$, there exists a local orthonormal frame*

$$\left\{\frac{1}{\sqrt{1-C^2}}V,\ (X_1,0),\dots,(X_{n-1},0),\ (0,Y_n),\dots,(0,Y_{n+m-2})\right\}$$

with respect to which the shape operator A of Σ satisfies

$$\begin{cases} AV &= 0,\\ \langle A(X_i,0),(X_j,0)\rangle &= \lambda_i\delta_{ij}, \quad i,j = 1,\dots,n-1,\\ \langle A(0,Y_\alpha),(0,Y_\beta)\rangle &= \lambda_\alpha\delta_{\alpha\beta}, \quad \alpha,\beta = n,\dots,n+m-2.\end{cases} \tag{3.1}$$

Here, λ_i/C_1 ($i = 1,\dots,n-1$) are the principal curvatures of Σ_y in M_1^n, and λ_α/C_2 ($\alpha = n,\dots,n+m-2$) are the principal curvatures of Σ_x in M_2^m. Moreover, the mean curvature of Σ is given by $H = \sum_{i=1}^{n+m-2}\lambda_i$.

Proof. Denote by ∇, ∇^h, and ∇^v the Levi-Civita connections on $M_1^n \times M_2^m$, M_1^n, and M_2^m, respectively. Since the hypersurface Σ has a constant angle function C, for any $X \in \mathfrak{X}(\Sigma_y)$ and $Y \in \mathfrak{X}(\Sigma_x)$, we have

$$\begin{aligned}\langle \nabla^h_X N^h, N^h\rangle_{M_1} &= \frac{1}{2} X\langle N^h, N^h\rangle_{M_1} = 0,\\ \langle \nabla^v_Y N^v, N^v\rangle_{M_2} &= \frac{1}{2} Y\langle N^v, N^v\rangle_{M_2} = 0.\end{aligned}$$

Hence, the shape operators A_{N^h} of $\Sigma_y \subset M_1^n$ and A_{N^v} of $\Sigma_x \subset M_2^m$ satisfy

$$\begin{aligned}C_1 A_{N^h}X &= -\nabla^h_X N^h + \langle \nabla^h_X N^h, N^h\rangle_{M_1} N^h = -\nabla^h_X N^h,\\ C_2 A_{N^v}Y &= -\nabla^v_Y N^v + \langle \nabla^v_Y N^v, N^v\rangle_{M_2} N^v = -\nabla^v_Y N^v.\end{aligned}$$

At each point $(x, y) \in \Sigma$, let $\{(X_i, 0)\}_{i=1}^{n-1}$ be eigenvectors of A_{N^h} corresponding to eigenvalues λ_i/C_1, and $\{(0, Y_\alpha)\}_{\alpha=n}^{n+m-2}$ be eigenvectors of A_{N^v} corresponding to eigenvalues λ_α/C_2, respectively. Then we have

$$\begin{aligned}\langle A(X_i, 0), (X_j, 0)\rangle &= -\langle \nabla_{(X_i,0)}(N^h, N^v), (X_j, 0)\rangle = -\langle \nabla^h_{X_i} N^h, X_j\rangle_{M_1^n} \\ &= \langle C_1 A_{N^h} X_i, X_j\rangle_{M_1^n} = \lambda_i \delta_{ij}\end{aligned}$$

for any $i, j = 1, \ldots, n-1$ and similarly,

$$\begin{aligned}\langle A(0, Y_\alpha), (0, Y_\beta)\rangle &= -\langle \nabla_{(0,Y_\alpha)}(N^h, N^v), (0, Y_\beta)\rangle = -\langle \nabla^v_{Y_\alpha} N^v, Y_\beta\rangle_{M_2^m} \\ &= \langle C_2 A_{N^v} Y_\alpha, Y_\beta\rangle_{M_2^m} = \lambda_\alpha \delta_{\alpha\beta}\end{aligned}$$

for any $\alpha, \beta = n, \ldots, n+m-2$. Therefore, equation (3.1) follows. □

Now, we proceed to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. According to Theorem 1.1, the isoparametric hypersurface Σ possesses a constant angle function C. The cases $C = 1$ and $C = -1$ correspond to (i) and (ii), respectively; hence we assume $-1 < C < 1$ in the sequel.

We first consider the case $n = 1$, i.e., $\mathbb{S}^1 \times \mathbb{R}^m$, which leads to parts (i)–(iii) of the classification. Recall the following result from [20].

Lemma 3.4 ([20])**.** *Let $\pi : E \to B$ be a Riemannian submersion with minimal fibers. Given any (properly) isoparametric function f on B, then $F := f \circ \pi$ is a (properly) isoparametric function on E.*

The universal cover $\pi : \mathbb{R} \to \mathbb{S}^1$, $\pi(x) = e^{\sqrt{-1}x}$, has discrete (hence minimal) fibers, and the induced covering map

$$\begin{aligned}\tilde{\pi} : \mathbb{R}^{m+1} &\longrightarrow \mathbb{S}^1 \times \mathbb{R}^m \\ (x, y) &\longmapsto (\pi(x), y)\end{aligned}$$

is a Riemannian submersion with minimal fibers. By Lemma 3.4, it suffices to find an isoparametric function F on $\mathbb{R}^{m+1}$ satisfying $F = f \circ \tilde{\pi}$, where f is an isoparametric function on $\mathbb{S}^1 \times \mathbb{R}^m$. Notice that the periodicity of $\tilde{\pi}$ implies that $F(x+2k\pi, y) = F(x, y)$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$.

If the foliation determined by F admits a focal manifold Σ_0, the classification in $\mathbb{R}^{m+1}$ implies that Σ_0 is either a single point or an affine subspace of dimension at most $m-1$. Moreover, for any $(x, y) \in \Sigma_0$, the entire line $\mathbb{R} \times \{y\} \subset \Sigma_0$; otherwise Σ_0 would decompose into disjoint union of lower-dimensional affine subspaces, which does not occur in the classification. Consequently, $F(x + 2k\pi, y) = F(x, y)$ for any $y \in \mathbb{R}^m$, and the identity $F(x, y) = F(x', y)$ holds for all $x, x' \in \mathbb{R}$, thereby proving Theorem 1.4-(ii).

If F admits no focal manifold, the classification in $\mathbb{R}^{m+1}$ implies that its regular level sets must be hyperplanes. The periodicity condition allows one to choose $F(x, y) = \sin(x - \kappa\langle y, y_0\rangle)$, where y_0 is a unit vector in $\mathbb{R}^m$. When $\kappa = 0$, a connected component of the regular level set of F corresponds to Theorem 1.4-(i); when $\kappa \neq 0$, each connected component of a regular level set of F can be parameterized as in Theorem 1.4-(iii).

Next, consider $n \geq 2$, which leads to parts (i), (ii), and (iv). The constancy of C implies that C_1 and C_2 are also constant. Denote by F the isoparametric function on $M_c^n \times \mathbb{R}^m$ associated with Σ.

Case 1: $\mathbb{S}^n \times \mathbb{R}^m$ $(n \geq 2)$. For any $(x, y) \in \Sigma$, Proposition 3.1 shows that Σ_x and Σ_y are regular level sets of transnormal functions on $\mathbb{R}^m$ and $\mathbb{S}^n$, respectively, and hence are isoparametric by [26, Theorem 1.5-(1)]. However, isoparametric hypersurfaces in $\mathbb{S}^n$ have focal points that occur infinitely often along each normal geodesic. Using

$$\exp_{(x,y)} tN = \left(x \cos C_1 t + \frac{\sin C_1 t}{C_1} N^h,\; y + tN^v \right),$$

it follows that Σ_x would have infinitely many focal points in $\mathbb{R}^m$, contradicting their classification. Therefore, no isoparametric hypersurfaces with $-1 < C < 1$ exist in $\mathbb{S}^n \times \mathbb{R}^m$.

Case 2: $\mathbb{H}^n \times \mathbb{R}^m$. By [26, Theorem 1.1], the possible topological types of $\mathbb{H}^n \times \mathbb{R}^m$ are as follows:

(i) If the transnormal system has no focal submanifold: an $\mathbb{R}$-bundle or $\mathbb{S}^1$-bundle over a hypersurface Σ.

(ii) If there is one focal submanifold: either a vector bundle over the unique focal submanifold $\widetilde{\Sigma}$ or a DDBD structure.

(iii) If there are two focal submanifolds: a DDBD structure.

where DDBD (Double Disc Bundle Decomposition) structure means that the ambient manifold is constructed by glueing two disc bundles over two submanifolds along the boundaries.

The $\mathbb{S}^1$-bundle case is excluded since $\exp_{(x,y)} tN \neq (x, y)$ for any $t \neq 0$.

The DDBD structure is also impossible. If $\mathbb{H}^n \times \mathbb{R}^m$ admitted a DDBD structure, then for any point $(x, y) \in \Sigma$, the normal geodesic would intersect the focal manifold infinitely many times, yielding infinitely many focal points along it. By Proposition 3.1, this implies that Σ_x also has infinitely many focal points along the normal geodesic in $\mathbb{R}^m$, contradicting the known focal structure of isoparametric hypersurfaces in $\mathbb{R}^m$.

As for the remaining two cases, we first show that Σ_y is an isoparametric hypersurface in $\mathbb{H}^n$, since the isoparametricity of Σ_x in $\mathbb{R}^m$ follows from a similar discussion as in Case 1. Furthermore, we will see that Σ_x is isometric to $\Sigma_{x'}$ and Σ_y is isometric to $\Sigma_{y'}$ for any $(x, y), (x', y') \in \Sigma$.

In case $\mathbb{H}^n \times \mathbb{R}^m$ is a vector bundle over its unique focal submanifold $\widetilde{\Sigma}$, let $\Sigma \subset \mathbb{H}^n \times \mathbb{R}^m$ be the tube of constant radius t around $\widetilde{\Sigma}$. For any $(x, y) \in \Sigma$, we have

$$\exp_{(x,y)} \tfrac{2t}{1-C}(0, N^v) = \exp_{\exp_{(x,y)}(-\frac{t}{1-C}V)} tN,$$
$$\exp_{(x,y)} \tfrac{2t}{1+C}(N^h, 0) = \exp_{\exp_{(x,y)}(\frac{t}{1+C}V)} tN.$$

Hence $\Sigma_x \subset \mathbb{R}^m$ lies at distance $\frac{t}{C_2}$ from its focal submanifold along the unit normal N^v/C_2, and $\Sigma_y \subset \mathbb{H}^n$ lies at distance $\frac{t}{C_1}$ along N^h/C_1. Moreover, Σ_x has constant mean curvature $H_{\Sigma_x} = \ell C_2/t$ for some integer $\ell \in \{1, \ldots, m-1\}$. Using Proposition 3.3, since Σ has a constant angle function $C \in (-1, 1)$,

$$H_\Sigma(x, y) = C_1 H_{\Sigma_y}(x) + C_2 H_{\Sigma_x}(y),$$

and since H_Σ, H_{Σ_x}, C_1 and C_2 are constant, so is H_{Σ_y}. Let Σ_t denote the parallel hypersurface at distance t from Σ, and $\Sigma_{y,t}$ the parallel hypersurface at distance t from

$\Sigma_y \subset \mathbb{H}^n$. Noting that

$$\exp_{(x,y)} \frac{2}{1+C} t(N^h, 0) = \exp_{\exp_{(x,y)} \frac{1}{1+C} tV} tN,$$

we obtain

$$H_{\Sigma_{C_1 t}}(x,y) = C_1 H_{\Sigma_{y,t}}\Big(\exp_x t\frac{N^h}{C_1}\Big) + C_2 H_{\Sigma_{\exp_x t\frac{N^h}{C_1}}}(y).$$

Since $H_{\Sigma_{C_1 t}}$ and $H_{\Sigma_{\exp_x t\frac{N^h}{C_1}}}$ are both constant, so is $H_{\Sigma_{y,t}}$. Hence Σ_y is an isoparametric hypersurface in $\mathbb{H}^n$. Moreover, since an isoparametric hypersurface with a single focal submanifold in $\mathbb{H}^n$ or $\mathbb{R}^m$ is uniquely determined (up to isometry) by its distance to the focal submanifold, it follows that Σ_x is isometric to $\Sigma_{x'}$ and Σ_y to $\Sigma_{y'}$ for any $(x,y),(x',y') \in \Sigma$.

In case $\mathbb{H}^n \times \mathbb{R}^m$ is an $\mathbb{R}$-bundle over the hypersurface Σ, in this case, $\Sigma \subset \mathbb{R}^m$ has no focal points. It follows that Σ_x also has no focal points, thus are hyperplanes with vanishing mean curvatures. Then an analogous discussion as the previous case shows that $\Sigma_y \subset \mathbb{H}^n$ is isoparametric. Moreover, for any $(x,y) \in \Sigma$, $\Sigma_y \subset \mathbb{H}^n$ is one of the following:

(i) a totally geodesic hyperplane ($\lambda_i = 0$),

(ii) an equidistant hypersurface ($0 < |\lambda_i| < C_1$), or

(iii) a horosphere ($\lambda_i = \pm C_1$),

while $\Sigma_x \subset \mathbb{R}^m$ is a hyperplane ($\lambda_\alpha = 0$). In these cases, principal curvatures at $(x,y),(x',y') \in \Sigma$ are the same. Since isoparametric hypersurfaces in $\mathbb{H}^n$ and $\mathbb{R}^m$ without focal submanifolds are uniquely determined by their principal curvatures, we again conclude that Σ_x and Σ_y are pairwise isometric.

Next, we consider the flow along V:

$$\exp_{(x,y)} tV = \Big(\exp_x\Big((1-C)tN^h\Big), \exp_y\Big(-(1+C)tN^v\Big)\Big).$$

From Proposition 3.3, we have

$$\begin{aligned} A(X_i,0) &= \lambda_H^i(X_i,0) + \sigma_{i\alpha}(0,Y_\alpha),\\ A(0,Y_\alpha) &= \sigma_{\alpha i}(X_i,0) + \lambda_R^\alpha(0,Y_\alpha), \end{aligned}$$

where $(\sigma_{i\alpha})$ is an $(n-1)\times(m-1)$ matrix. Let A_t denote the shape operator at $\exp_{(x,y)} tV$. Since Σ_y and Σ_x are isometric to $\Sigma_{\exp_y(-(1+C)tN^v)}$ and $\Sigma_{\exp_x((1-C)tN^h)}$, respectively, we may assume

$$\begin{aligned} A_t(X_i,0) &= p_{ki}\lambda_H^k p_{kj}(X_j,0) + p_{ki}\sigma_{k\alpha}q_{\alpha\beta}(0,Y_\beta),\\ A_t(0,Y_\alpha) &= q_{\gamma\alpha}\sigma_{i\gamma}p_{ij}(X_j,0) + q_{\gamma\alpha}\lambda_R^\gamma q_{\gamma\beta}(0,Y_\beta), \end{aligned}$$

where (p_{ij}) and $(q_{\alpha\beta})$ are orthogonal matrices of orders $n-1$ and $m-1$. Differentiating $A_t\Big((f_t)_*(X,Y)\Big) = -\nabla_{(f_t)_*(X,Y)}N$ at $t=0$ yields

$$\sum_i (\lambda_H^i)^2 = (n-1)C_1^2 + \frac{(1+C)^2}{(1-C)^2}\sum_\alpha (\lambda_R^\alpha)^2. \tag{3.2}$$

If Σ_y and Σ_x focalize simultaneously, we can view Σ as a tube of radius s around the focal submanifold, giving

$$\lambda_H^1 = \cdots = \lambda_H^k = C_1 \coth \frac{s}{C_1}, \qquad \lambda_H^{k+1} = \cdots = \lambda_H^{n-1} = C_1 \tanh \frac{s}{C_1},$$
$$\lambda_R^1 = \cdots = \lambda_R^\ell = \frac{C_2^2}{s}, \qquad \lambda_R^{\ell+1} = \cdots = \lambda_R^{m-1} = 0,$$

for some $k \in \{1, \ldots, n-1\}$ and $\ell \in \{1, \ldots, m-1\}$. However, equation (3.2) contradicts the above equations for all s.

If Σ has no focal points, the classification implies that Σ_x is a hyperplane in $\mathbb{R}^m$, i.e., $\lambda_R^i = 0$. Substituting this into (3.2), we obtain that $\lambda_H^1 = \cdots = \lambda_H^{n-1} = \pm C_1$. Therefore, the only remaining case is that Σ_x is a hyperplane in $\mathbb{R}^m$ and Σ_y is a horosphere in $\mathbb{H}^n$ for any $(x, y) \in \Sigma$, with $\sigma_{i\alpha} = 0$, leading directly to the expression in Theorem 1.4-(iv).

□

Example 3.5. From the proof of Theorem 1.4, the hypersurfaces described in case (iii) arise as connected components of the level sets of

$$F : \mathbb{S}^1 \times \mathbb{R}^m \to \mathbb{R}, \qquad F(e^{\sqrt{-1}x}, y) = \sin\Big(x - \kappa\langle y, y_0\rangle\Big),$$

where y_0 is a fixed unit vector in $\mathbb{R}^m$ and $\kappa \in \mathbb{R}$.

A direct computation yields

$$\nabla F = \Big(\cos(x - \kappa\langle y, y_0\rangle),\ -\kappa \cos(x - \kappa\langle y, y_0\rangle) y_0\Big),$$

and hence

$$\|\nabla F\|^2 = (1 + \kappa^2)(1 - F^2), \qquad \Delta F = -(1 + \kappa^2)F.$$

Thus, F is an isoparametric function on $\mathbb{S}^1 \times \mathbb{R}^m$.

All values of F except ± 1 are regular. Moreover, $F^{-1}(\pm 1)$ are also connected isoparametric hypersurfaces parameterized as in Theorem 1.4-(iii). For each $t \in (-1, 1)$, $F^{-1}(t)$ consists of two connected components.

Consider $\Sigma = F^{-1}(t)$ for $t \in [-1, 1]$. Its unit normal vector is

$$N = \mathrm{sgn}\Big(\cos(x - \kappa\langle y, y_0\rangle)\Big) \frac{1}{\sqrt{1 + \kappa^2}} (1, -\kappa y_0),$$

and the angle function is $C = \frac{1-\kappa^2}{1+\kappa^2}$. A straightforward computation shows that the Hessian of F satisfies $\nabla^2 F|_\Sigma = 0$, implying that Σ is totally geodesic in $\mathbb{S}^1 \times \mathbb{R}^m$.

Indeed,

$$\nabla^2 F = \sin\Big(x - \kappa\langle y, y_0\rangle\Big) \begin{pmatrix} -1 & \kappa y_0^T \\ \kappa y_0 & -\kappa^2 y_0 y_0^T \end{pmatrix}.$$

Choose an orthonormal frame $\{v_1, \ldots, v_m\}$ on Σ, where $v_i = (0, Y_i)$ for $i = 1, \ldots, m-1$ and

$$v_m = \frac{V}{\|V\|} = \mathrm{sgn}\Big(\cos(x - \kappa\langle y, y_0\rangle)\Big)\Big(\tfrac{|\kappa|}{\sqrt{1+\kappa^2}},\ \tfrac{\mathrm{sgn}(\kappa)}{\sqrt{1+\kappa^2}} y_0\Big).$$

For $i, j \le m - 1$, since $Y_j \perp y_0$ and $\|y_0\| = 1$, one verifies

$$\nabla^2 F|_\Sigma(v_i, v_j) = 0, \qquad \nabla^2 F|_\Sigma(v_i, v_m) = 0,$$

and hence $\nabla^2 F|_\Sigma = 0$.

We now show that each connected component of a level set of F is homogeneous in $\mathbb{S}^1 \times \mathbb{R}^m$. When $\kappa = 0$, $F = \sin x$, corresponding to case (i) of Theorem 1.4. Hence we assume $\kappa \neq 0$.

Let Σ_0 be a connected component of $\Sigma = F^{-1}(t)$ for $t \neq \pm 1$. Denote by $\mathrm{Isom}_0(\mathbb{S}^1 \times \mathbb{R}^m)$ the identity component of the isometry group of $\mathbb{S}^1 \times \mathbb{R}^m$, which is isomorphic to $SO(2) \times (\mathbb{R}^m \rtimes SO(m))$, where $SO(2)$ is the special orthogonal group of degree 2 and $\mathbb{R}^m \rtimes SO(m)$ is the special Euclidean group in m dimensions. Represent $y \in \mathbb{R}^m$ by $\begin{pmatrix} y \\ 1 \end{pmatrix}$ and consider the subgroup

$$K = \langle K_1, K_2 \rangle \subset \mathrm{Isom}_0(\mathbb{S}^1 \times \mathbb{R}^m),$$

where

$$K_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & B & b \\ 0 & 0 & 1 \end{pmatrix} \middle| B \in SO(m),\ B^T y_0 = y_0,\ \langle b, y_0 \rangle = 0 \right\},$$

and

$$K_2 = \left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 & 0 \\ 0 & I_m & \frac{\theta}{\kappa} y_0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}.$$

Since K_1 and K_2 commute and $K_1 \cap K_2 = \{Id\}$, we have $K \cong K_1 \times K_2$, i.e.,

$$K = \left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 & 0 \\ 0 & B & b + \frac{\theta}{\kappa} y_0 \\ 0 & 0 & 1 \end{pmatrix} \middle| B^T y_0 = y_0,\ \langle b, y_0 \rangle = 0,\ B \in SO(m),\ \theta \in \mathbb{R} \right\}.$$

Define

$$\begin{aligned} \phi : \mathrm{Isom}_0(\mathbb{S}^1 \times \mathbb{R}^m) &\longrightarrow SO(2) \times \mathbb{R}^2, \\ \left(e^{\sqrt{-1}\theta}, \begin{pmatrix} B & b \\ 0 & 1 \end{pmatrix} \right) &\longmapsto \left(e^{\sqrt{-1}\theta}, \langle B^T y_0, y_0 \rangle, \langle b, y_0 \rangle \right). \end{aligned}$$

Evidently, ϕ is continuous. Since $K = \phi^{-1}(D)$ with $D = \{(e^{\sqrt{-1}\theta}, 1, \theta/\kappa) \mid \theta \in \mathbb{R}\}$ closed in $SO(2) \times \mathbb{R}^2$, K is a closed subgroup of $\mathrm{Isom}(\mathbb{S}^1 \times \mathbb{R}^m)$.

Finally, Σ_0 is an orbit of K. For $(x, y), (x', y') \in \Sigma_0$, we have $x' - x = \kappa \langle y' - y, y_0 \rangle$. Moreover, since y and $y' + \langle y - y', y_0 \rangle y_0$ lie in the same hyperplane perpendicular to y_0, we can choose $B \in SO(m)$ and $b \in \mathbb{R}^m$ such that $B^T y_0 = y_0$ and $By + b + \frac{x'-x}{\kappa} y_0 = y'$. Then

$$\begin{pmatrix} e^{\sqrt{-1}(x'-x)} & 0 & 0 \\ 0 & B & b + \frac{x'-x}{\kappa} y_0 \\ 0 & 0 & 1 \end{pmatrix} \in K$$

maps (x, y) to (x', y'). Thus, K acts transitively on Σ_0 and preserves it, proving that Σ_0 is a homogeneous hypersurface in $\mathbb{S}^1 \times \mathbb{R}^m$. The same argument applies when $t = \pm 1$.

□

Example 3.6. The isoparametric function corresponding to case (iv) of Theorem 1.4 is

$$F : \mathbb{H}^n \times \mathbb{R}^m \to \mathbb{R}, \qquad F(x, y) = \langle x, u \rangle_{\mathbb{L}} \exp\Big(a \langle y - y_0, v_0 \rangle \Big),$$

where $u = (u_0, \dots, u_n)$ is a nonzero lightlike vector in Lorentz space $\mathbb{L}^{n+1}$ with $u_0 > 0$, $\langle \cdot, \cdot \rangle_{\mathbb{L}}$ denotes the Lorentz inner product, v_0 is a fixed unit vector in $\mathbb{R}^m$, $y_0 \in \mathbb{R}^m$, and $a \in \mathbb{R}$.

A direct computation gives

$$\nabla F = \Big((u + \langle x, u \rangle_{\mathbb{L}} x) \exp(a\langle y - y_0, v_0 \rangle),\ a v_0 \langle x, u \rangle_{\mathbb{L}} \exp(a\langle y - y_0, v_0 \rangle) \Big),$$

hence

$$\|\nabla F\|^2 = (1 + a^2) F^2, \qquad \Delta F = (n + a^2) F.$$

Thus, F is an isoparametric function, and all its level sets are regular.

For fixed $y \in \mathbb{R}^m$, the equation $F(x, y) = t$ gives $\langle x, u \rangle_{\mathbb{L}} = t \exp(-a\langle y - y_0, v_0 \rangle)$, representing a horosphere in $\mathbb{H}^n$ centered at the lightlike vector u. For fixed $x \in \mathbb{H}^n$, one obtains $\langle y - y_0, v_0 \rangle = \frac{1}{a} \ln \frac{t}{\langle x, u \rangle_{\mathbb{L}}}$, defining an affine hyperplane in $\mathbb{R}^m$ through y_0 with unit normal v_0.

Let $\Sigma = F^{-1}(t)$ for $t \in (-\infty, 0)$. Its unit normal and angle function are

$$N = \frac{1}{\sqrt{1 + a^2}} \Big(\frac{u + \langle x, u \rangle_{\mathbb{L}} x}{\langle x, u \rangle_{\mathbb{L}}},\ a v_0 \Big), \qquad C = \frac{1 - a^2}{1 + a^2}.$$

The cases $a = 0$ and $|a| \to \infty$ correspond to Theorem 1.4-(i) and -(ii), respectively; thus we focus on $a \neq 0$.

For tangent vectors $X = (X^h, X^v)$ and $Y = (Y^h, Y^v)$ of $\mathbb{H}^n \times \mathbb{R}^m$, the Hessian of F is

$$\begin{aligned}\nabla^2 F(X, Y) =& \langle X^h, Y^h \rangle F + a^2 \langle X^v, v_0 \rangle \langle Y^v, v_0 \rangle F \\ &+ a \exp\Big(a\langle y - y_0, v_0 \rangle \Big) \Big(\langle X^h, u^\top \rangle \langle Y^v, v_0 \rangle + \langle Y^h, u^\top \rangle \langle X^v, v_0 \rangle \Big),\end{aligned}$$

where $u^\top = u + \langle u, x \rangle_{\mathbb{L}} x$ is the projection of u onto $T_x \mathbb{H}^n$.

Choose an orthonormal frame $\{(X_1, 0), \dots, (X_{n-1}, 0), (0, Y_1), \dots, (0, Y_{m-1}), V/\|V\|\}$ on Σ, where $V = PN - CN$ and

$$\frac{V}{\|V\|} = \frac{1}{\sqrt{1 + a^2}} \Big(|a| \frac{u + \langle x, u \rangle_{\mathbb{L}} x}{\langle x, u \rangle_{\mathbb{L}}},\ -\mathrm{sgn}(a) v_0 \Big).$$

Under this frame,

$$\nabla^2 F|_\Sigma = \begin{pmatrix} t\, I_{n-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the second fundamental form $\mathrm{II} = -\frac{1}{\|\nabla F\|} \nabla^2 F|_\Sigma$, the principal distributions are

$$\begin{aligned} \mathcal{V}_1 &= \mathrm{span}\, \{(X_i, 0) \mid i = 1, \dots, n-1\}, \\ \mathcal{V}_2 &= \mathrm{span}\, \{(0, Y_j) \mid j = 1, \dots, m-1\}, \\ \mathcal{V}_3 &= \mathrm{span}\, \{V\}. \end{aligned}$$

with corresponding principal curvatures and multiplicities:

Distribution	Principal curvature	Multiplicity
$\mathcal{V}_1$	$\frac{1}{\sqrt{1 + a^2}}$	$n - 1$
$\mathcal{V}_2$	0	$m - 1$
$\mathcal{V}_3$	0	1

Hence the mean curvature of Σ is $H = \dfrac{n-1}{\sqrt{1+a^2}}$. For any principal directions X, Y, the sectional curvature is

$$K_\Sigma(X,Y) = K_{\mathbb{H}^n\times\mathbb{R}^m}(X,Y) + \frac{\mathrm{II}(X,X)\mathrm{II}(Y,Y) - \mathrm{II}(X,Y)^2}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2},$$

yielding the following table of sectional curvatures:

$X \in$ \ $Y \in$	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$
$\mathcal{V}_1$	$-\dfrac{a^2}{1+a^2}$	0	$-\dfrac{a^2}{1+a^2}$
$\mathcal{V}_2$	0	0	0
$\mathcal{V}_3$	$-\dfrac{a^2}{1+a^2}$	0	$-$

Thus the Ricci and scalar curvatures are

$$\mathrm{Ric}_\Sigma X = \begin{cases} -\dfrac{(n-1)a^2}{1+a^2}, & X \in \mathcal{V}_1 \cup \mathcal{V}_3, \\ \quad 0, & X \in \mathcal{V}_2, \end{cases} \qquad R = -\frac{n(n-1)a^2}{1+a^2}.$$

Clearly, Σ is not Einstein when $a \neq 0$.

To show Σ is homogeneous, define the subgroup

$$G = \langle G_1, G_2\rangle \subset \mathrm{Isom}_0(\mathbb{H}^n \times \mathbb{R}^m) \cong SO^+(1,n) \times (\mathbb{R}^m \rtimes SO(m)),$$

where $SO^+(1,n)$ denotes the identity component of the Lorentz group and

$$G_1 = \left\{ \begin{pmatrix} B & 0 & 0 \\ 0 & I_m & sv_0 \\ 0 & 0 & 1 \end{pmatrix} \middle|\, B \in SO^+(1,n),\ B^T u = e^{-as}u,\ s \in \mathbb{R} \right\},$$

$$G_2 = \left\{ \begin{pmatrix} I_n & 0 & 0 \\ 0 & \widetilde{B} & b \\ 0 & 0 & 1 \end{pmatrix} \middle|\, \widetilde{B} \in SO(m),\ \widetilde{B}^T v_0 = v_0,\ \langle b, v_0\rangle = 0 \right\}.$$

An analogous discussion as in Example 3.5 shows that G_1 and G_2 commute, and thus $G \cong G_1 \times G_2$, i.e.,

$$G = \left\{ \begin{pmatrix} B & 0 & 0 \\ 0 & \widetilde{B} & b + sv_0 \\ 0 & 0 & 1 \end{pmatrix} \middle|\, \begin{matrix} B^T u = e^{-as}u,\ B \in SO^+(1,n),\ s \in \mathbb{R}, \\ \widetilde{B}^T v_0 = v_0,\ \langle b, v_0\rangle = 0,\ \widetilde{B} \in SO(m) \end{matrix} \right\}.$$

Define

$$\eta : \mathrm{Isom}_0(\mathbb{H}^n \times \mathbb{R}^m) \to \mathbb{R}^3, \quad \left(B, \begin{pmatrix} \widetilde{B} & \widetilde{b} \\ 0 & 1 \end{pmatrix}\right) \mapsto \Big(\langle B^T u, u\rangle_{\mathbb{L}}, \langle \widetilde{B}^T v_0, v_0\rangle, \langle \widetilde{b}, v_0\rangle\Big).$$

Evidently, η is continuous. Then $G = \eta^{-1}(D)$ with $D = \{(e^{-as}, 1, s) \mid s \in \mathbb{R}\}$ closed in $\mathbb{R}^3$; hence G is a closed subgroup of $\mathrm{Isom}_0(\mathbb{H}^n \times \mathbb{R}^m)$.

Finally, for $(x,y), (x',y') \in \Sigma = F^{-1}(t) \subset \mathbb{H}^n \times \mathbb{R}^m$, $\langle x', u\rangle_{\mathbb{L}} = \langle x, u\rangle_{\mathbb{L}} \exp\Big(-a\langle y' - y, v_0\rangle\Big)$. Then the transitivity of the isometric $SO^+(1,n)$-action on $\mathbb{H}^n$ yields the existence of $B_0 \in SO^+(1,n)$ such that $B_0 x = x'$, and thus

$$\langle B_0 x, u\rangle_{\mathbb{L}} = \langle x, B_0^T u\rangle_{\mathbb{L}} = \langle x, u\rangle_{\mathbb{L}} \exp\Big(-a\langle y' - y, v_0\rangle\Big),$$

which implies $B_0^T u = \exp\Big(-a\langle y'-y, v_0\rangle\Big)u$. Similarly, there exist $\widetilde{B}_0 \in SO(m)$ and $b_0 \in \mathbb{R}^m$ such that $\widetilde{B}_0 y + b_0 + \langle y'-y, v_0\rangle v_0 = y'$, $\langle b_0, v_0\rangle = 0$, thus $\widetilde{B}_0 v_0 = v_0$. Then

$$g = \begin{pmatrix} B_0 & 0 & 0 \\ 0 & \widetilde{B}_0 & b_0 + \langle y'-y, v_0\rangle v_0 \\ 0 & 0 & 1 \end{pmatrix} \in G$$

maps (x, y) to (x', y'). Thus G acts transitively on Σ, and since F is G-invariant, G preserves Σ. Therefore, Σ is a homogeneous hypersurface in $\mathbb{H}^n \times \mathbb{R}^m$.

□

4. Proof of Theorem 1.6

Proposition 4.1. *Let Σ be an orientable hypersurface in $M^n_{c_1} \times M^m_{c_2}$. Then*

(i) *for any $(X, Y) \in \mathfrak{X}(\Sigma)$, the covariant derivative of V is given by*

$$\nabla^{\Sigma}_{(X,Y)} V = CA(X, Y) - P^{\top} A(X, Y), \tag{4.1}$$

where $P^{\top} : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the tangential projection of P onto Σ;

(ii) *if Σ has constant angle and constant principal curvatures, then for any principal vector field (X, Y) orthogonal to V, one has*

$$\nabla^{\Sigma}_V (X, Y) = 0. \tag{4.2}$$

Proof. (i) Recall that P is parallel and that $V = PN - CN$. Then we have

$$\nabla_{(X,Y)} V = P\nabla_{(X,Y)} N - \Big(\nabla_{(X,Y)} C\Big) N - C\nabla_{(X,Y)} N.$$

By taking the tangential component of this expression, we obtain equation (4.1). When $M^n_{c_1} \times M^m_{c_2} = \mathbb{S}^2 \times \mathbb{S}^2$, the result coincides with Lemma 1 in [41].

(ii) We consider the flow of the vector field V, denoted by $f_t : \Sigma \to \Sigma$, which is defined by $f_t(x, y) = \exp_{(x,y)}(tV)$. Let A be the shape operator at the point $(x, y) \in \Sigma$, and A_t the shape operator at $f_t(x, y)$. For convenience, we introduce the functions $\mathcal{C}_i(t)$ and $\mathcal{S}_i(t)$ $(i = 1, 2)$ as follows:

$$\mathcal{C}_i(t) = \begin{cases} \cos t, & c_i > 0, \\ 1, & c_i = 0, \\ \cosh t, & c_i < 0, \end{cases} \qquad \mathcal{S}_i(t) = \begin{cases} \sin t, & c_i > 0, \\ t, & c_i = 0, \\ \sinh t, & c_i < 0. \end{cases}$$

Assume that (X, Y) is a principal direction corresponding to the principal curvature λ, and that it is orthogonal to V. A straightforward computation shows that

$$\begin{aligned} (f_t)_*(X, Y) =& (X, Y) \begin{pmatrix} \mathcal{C}_1\left(C_1(1-C)t\right) & \\ & \mathcal{C}_2\left(C_2(1+C)t\right) \end{pmatrix} \\ & - A(X, Y) \begin{pmatrix} \frac{1}{C_1}\mathcal{S}_1\left(C_1(1-C)t\right) & \\ & -\frac{1}{C_2}\mathcal{S}_2\left(C_2(1+C)t\right) \end{pmatrix}. \end{aligned} \tag{4.3}$$

Differentiating this with respect to t at $t = 0$, we obtain

$$[V, (X, Y)] = -\frac{d}{dt}\bigg|_{t=0} (f_t)_*(X, Y) = \lambda((1-C)X, -(1+C)Y).$$

On the other hand, from equation (4.1) we have

$$\nabla^{\Sigma}_{(X,Y)} V = CA(X,Y) - P^{\top} A(X,Y) = \lambda\Big(-(1-C)X, (1+C)Y\Big).$$

Hence, it follows that

$$\nabla^{\Sigma}_{V}(X,Y) = [V,(X,Y)] + \nabla^{\Sigma}_{(X,Y)} V = 0.$$

$\square$

Now, we proceed to complete the proof of Theorem 1.6.

Proof of Theorem 1.6.

When $C = 1$, we have $N = (N^h, 0)$, and hence Σ reduces to the product $\Sigma_1 \times M^m_{c_2}$, where Σ_1 is a hypersurface in $M^n_{c_1}$ with constant principal curvatures. The desired conclusion follows immediately. The case $C = -1$ is completely analogous and will be omitted. From now on, we focus on the case $-1 < C < 1$, that is, $C_1, C_2 \neq 0$.

In the flat case $c_1 = c_2 = 0$, namely $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$, the statement holds trivially. Thus, we only need to consider the case $c_1^2 + c_2^2 > 0$.

We employ the same notations and computations as in the proof of Proposition 4.1-(ii). Under the assumption that Σ has a constant angle function C, we observe from (2.3) that $AV = 0$. Assume that (X,Y) is a principal direction corresponding to the principal curvature λ and orthogonal to V. From $A_t\Big((f_t)_*(X,Y)\Big) = -\nabla_{(f_t)_*(X,Y)} N$, it follows that

$$\begin{aligned}
&A_t\Bigg((X,Y)\begin{pmatrix} \mathcal{C}_1\left(C_1(1-C)t\right) & \\ & \mathcal{C}_2\left(C_2(1+C)t\right)\end{pmatrix}\\
&\qquad\qquad -A(X,Y)\begin{pmatrix} \frac{1}{C_1}\mathcal{S}_1\left(C_1(1-C)t\right) & \\ & -\frac{1}{C_2}\mathcal{S}_2\left(C_2(1+C)t\right)\end{pmatrix}\Bigg)\\
=&-(X,Y)\begin{pmatrix} -c_1C_1\mathcal{S}_1\left(C_1(1-C)t\right) & \\ & c_2C_2\mathcal{S}_2\left(C_2(1+C)t\right)\end{pmatrix}\\
&+A(X,Y)\begin{pmatrix} \mathcal{C}_1\left(C_1(1-C)t\right) & \\ & \mathcal{C}_2\left(C_2(1+C)t\right)\end{pmatrix}.
\end{aligned}\tag{4.4}$$

We distinguish the following two possibilities:

(1) There exists a principal direction (X,Y) orthogonal to V with $X \neq 0$ and $Y \neq 0$;

(2) No such direction exists.

Case (1). The equation (4.4) can be rewritten as

$$R_1(t)A_t(X,0) + R_2(t)A_t(0,Y) = -\frac{1}{1-C}R_1'(t)(X,0) + \frac{1}{1+C}R_2'(t)(0,Y),$$

where

$$\begin{aligned}
R_1(t) &= \mathcal{C}_1\left(C_1(1-C)t\right) - \frac{\lambda}{C_1}\mathcal{S}_1\left(C_1(1-C)t\right),\\
R_2(t) &= \mathcal{C}_2\left(C_2(1+C)t\right) + \frac{\lambda}{C_2}\mathcal{S}_2\left(C_2(1+C)t\right).
\end{aligned}\tag{4.5}$$

By Proposition 4.1-(ii), the shape operator A is invariant along the direction of V, i.e., $A_t = A$. Thus, we may write

$$\alpha_1\langle X,X\rangle = \langle A_t(X,0),(X,0)\rangle, \qquad \alpha_2\langle Y,Y\rangle = \langle A_t(X,0),(0,Y)\rangle, \tag{4.6}$$

$$\beta_1\langle X,X\rangle = \langle A_t(0,Y),(X,0)\rangle, \qquad \beta_2\langle Y,Y\rangle = \langle A_t(0,Y),(0,Y)\rangle, \tag{4.7}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ for $i = 1,2$.

Substituting these into (4.4) and comparing coefficients, we obtain

$$R_1(t)\alpha_1 + R_2(t)\beta_1 = -\frac{1}{1-C}R_1'(t), \tag{4.8}$$

$$R_1(t)\alpha_2 + R_2(t)\beta_2 = \frac{1}{1+C}R_2'(t). \tag{4.9}$$

We now discuss equations (4.8)–(4.9) for different values of c_i $(i = 1,2)$.

Case A: $c_1 \neq c_2$. In this situation, it follows directly that $\alpha_2 = \beta_1 = 0$, and the equations (4.8) and (4.9) simplifies to

$$R_1(t)\alpha_1 = -\frac{1}{1-C}R_1'(t), \qquad R_2(t)\beta_2 = \frac{1}{1+C}R_2'(t). \tag{4.10}$$

Evaluating at $t = 0$ gives $\alpha_1 = \beta_2 = \lambda$. Substituting equation (4.5) into equations (4.10) yields

$$\frac{\lambda^2 + c_1C_1^2}{C_1}\mathcal{S}_1(C_1(1-C)t) = 0, \qquad \frac{\lambda^2 + c_2C_2^2}{C_2}\mathcal{S}_2(C_2(1+C)t) = 0. \tag{4.11}$$

Since $C_1 \neq 0$, for any possible values of pair (c_1, c_2), none of the terms in (4.11) can vanish identically unless trivial or contradictory conditions occur. Therefore, Case A cannot occur within **Case (1).**

Case B: $c_1 = c_2 = c \neq 0$. Differentiating equations (4.8) and (4.9) at $t = 0$, and using $C_1^2 = \frac{1+C}{2}$ and $C_2^2 = \frac{1-C}{2}$, we obtain the following equalities from the first and second derivatives:

$$\lambda^2(1+C) - cC_1^2(1-C) = 2\lambda\alpha_1, \tag{4.12}$$

$$(\lambda^2 + cC_2^2)(1+C) = 2\lambda\alpha_2, \tag{4.13}$$

$$C(1-C^2)\alpha_1 = \lambda C(1-C^2), \tag{4.14}$$

$$C(1-C^2)\alpha_2 = 0. \tag{4.15}$$

Under the assumption $-1 < C < 1$, if $C \neq 0$, then (4.14)-(4.15) imply $\alpha_1 = \lambda$ and $\alpha_2 = 0$. Substituting these into (4.12)- (4.13) yields $C = 0$, a contradiction. Hence $C = 0$ in this case.

With $C = 0$, consider the parallel hypersurfaces $\Sigma_t = g_t(\Sigma)$ of Σ given by the immersion $g_t : \Sigma \to M_c^n \times M_c^m$, $g_t(x,y) = \exp_{(x,y)} tN$. For simplicity, write $\mathcal{C}(t) = \mathcal{C}_1(t) = \mathcal{C}_2(t)$ and $\mathcal{S}(t) = \mathcal{S}_1(t) = \mathcal{S}_2(t)$. Then it follows from $A_t\big((g_t)_*(X,Y)\big) = -\nabla_{(g_t)_*(X,Y)}N$ that

$$\begin{aligned}
&A_t\left((X,Y)\begin{pmatrix} \mathcal{C}(C_1t) & \\ & \mathcal{C}(C_2t)\end{pmatrix} - A(X,Y)\begin{pmatrix} \frac{1}{C_1}\mathcal{S}(C_1t) & \\ & \frac{1}{C_2}\mathcal{S}(C_2t)\end{pmatrix}\right)\\
&= -(X,Y)\begin{pmatrix} C_1\mathcal{S}(C_1t) & \\ & C_2\mathcal{S}(C_2t)\end{pmatrix} + A(X,Y)\begin{pmatrix} \mathcal{C}(C_1t) & \\ & \mathcal{C}(C_2t)\end{pmatrix}.
\end{aligned}$$

Using $C = 0$, we get

$$A_t(X,Y) = \frac{\frac{c}{\sqrt{2}}\mathcal{S}\left(\frac{t}{\sqrt{2}}\right) + \lambda\mathcal{C}\left(\frac{t}{\sqrt{2}}\right)}{\mathcal{C}\left(\frac{t}{\sqrt{2}}\right) - \sqrt{2}\lambda\mathcal{S}\left(\frac{t}{\sqrt{2}}\right)}(X,Y).$$

Since $AV = 0$ by (2.3), the mean curvature $H(t)$ of $\Sigma_t = g_t(\Sigma)$ is

$$H(t) = \sum_{i=1}^{n+m-2} \frac{\frac{c}{\sqrt{2}}\mathcal{S}\left(\frac{t}{\sqrt{2}}\right) + \lambda_i\mathcal{C}\left(\frac{t}{\sqrt{2}}\right)}{\mathcal{C}\left(\frac{t}{\sqrt{2}}\right) - \sqrt{2}\lambda_i\mathcal{S}\left(\frac{t}{\sqrt{2}}\right)}.$$

Therefore, Σ is an isoparametric hypersurface.

Case (2). The argument is parallel to Case (1). For any principal direction (X,Y) orthogonal to V, we obtain

$$\begin{aligned} &A_t\left((X,Y)\begin{pmatrix} \mathcal{C}_1(C_1t) & \\ & \mathcal{C}_2(C_2t)\end{pmatrix} - A(X,Y)\begin{pmatrix} \frac{1}{C_1}\mathcal{S}_1(C_1t) & \\ & \frac{1}{C_2}\mathcal{S}_2(C_2t)\end{pmatrix}\right) \\ =& -(X,Y)\begin{pmatrix} -c_1C_1\mathcal{S}_1(C_1t) & \\ & -c_2C_2\mathcal{S}_2(C_2t)\end{pmatrix} + A(X,Y)\begin{pmatrix} \mathcal{C}_1(C_1t) & \\ & \mathcal{C}_2(C_2t)\end{pmatrix}.\end{aligned}$$

Since Σ has only two types of principal directions apart from V, we find

$$A_t(X,0) = \frac{c_1C_1\mathcal{S}_1(C_1t) + \lambda\mathcal{C}_1(C_1t)}{\mathcal{C}_1(C_1t) - \frac{\lambda}{C_1}\mathcal{S}_1(C_1t)}(X,0).$$

$$A_t(0,Y) = \frac{c_2C_2\mathcal{S}_2(C_2t) + \lambda\mathcal{C}_2(C_2t)}{\mathcal{C}_2(C_2t) - \frac{\lambda}{C_2}\mathcal{S}_2(C_2t)}(0,Y).$$

Hence, each parallel hypersurface Σ_t has constant mean curvature, and thus Σ is isoparametric.

□

5. Proof of Theorem 1.1

As noted in Remark 1.3, we restrict attention to the case $n \geq 2$. Since the angle function is continuous, it suffices to show that it is locally constant. Hence, we consider only the case $-1 < C < 1$. For convenience, set $C_1 = \tau$. Clearly,

$$C_2 = \sqrt{1-\tau^2}, \qquad C = 2\tau^2 - 1, \qquad 0 < \tau < 1.$$

We will choose an orthonormal frame along the parallel hypersurface of Σ and compute the coefficient matrix of the Jacobi field with respect to this frame. Then, by analyzing the linear system satisfied by the mean curvature of the parallel hypersurface and its derivatives, we derive a nontrivial algebraic equation in τ, which in turn shows that the angle function C must be constant.

Let N_p denote the unit normal vector of Σ at $p \in \Sigma$, and define the normal exponential map $\Phi_r : \Sigma \to M_c^n \times \mathbb{R}^m$ by $\Phi_r(p) = \exp_p(rN_p)$. Then there exists a sufficiently small $\delta > 0$ such that, for all $r \in (-\delta, \delta)$, the map Φ_r is well defined and $\Sigma_r = \Phi_r(\Sigma)$ is an embedded hypersurface in $M_c^n \times \mathbb{R}^m$ at distance r from Σ. Fix $p \in \Sigma$, and let $\gamma_p(r)$, $r \in (-\delta, \delta)$, be the geodesic in $M_c^n \times \mathbb{R}^m$ satisfying $\gamma_p(0) = p$ and $\gamma_p'(0) = N_p$. The vector field $N(r) = \gamma_p'(r)$ along γ_p is parallel, and hence remains normal to Σ_r at $\gamma_p(r)$.

We now choose unit orthonormal vector fields $U_1(r), \dots, U_m(r)$ parallel along γ_p such that the horizontal components of $U_i(r)$ $(i = 1, \dots, m-1)$ vanish, i.e., $U_i^h(r) = 0$, and

$$U_m(r) = \left(-\frac{C_2}{C_1} N^h(r), \frac{C_1}{C_2} N^v(r)\right).$$

Together with $N(r)$, we extend these to obtain a unit orthonormal parallel frame

$$N(r),\ U_1(r),\ \dots,\ U_{n+m-1}(r)$$

along γ_p. By orthogonality, for $i = m+1, \dots, n+m-1$, the vector fields $U_i(r)$ have vanishing vertical components.

For each $j = 1, \dots, n+m-1$, let $\zeta_j(r)$ be the Jacobi field along γ_p satisfying

$$\zeta_j(0) = U_j(0), \qquad \zeta_j'(0) = -AU_j(0),$$

and

$$\zeta_j'' + R_c(\gamma_p', \zeta_j)\gamma_p' = 0, \tag{5.1}$$

where the Riemann curvature tensor R_c is defined in (2.4). To compute (5.1), we decompose $\zeta_j(r)$ in the orthonormal frame $\{U_i(r)\}_{i=1}^{n+m-1}$ as

$$\zeta_j(r) = \sum_{i=1}^{n+m-1} b_{ij}(r) U_i(r),$$

where $b_{ij}(r)$ are smooth functions on $(-\delta, \delta)$ for $j = 1, \dots, n+m-1$. Meanwhile, the shape operator A with respect to the orthonormal basis $\{U_i(0)\}_{i=1}^{n+m-1}$ is given by

$$AU_j(0) = \sum_{i=1}^{n+m-1} a_{ij} U_i(0).$$

We now decompose equation (5.1) into its horizontal and vertical components:

$$\zeta_j^{h\prime\prime} + R_c^h(\gamma'^h, \zeta_j^h)\gamma'^h = 0, \qquad \zeta_j^{v\prime\prime} + R_c^v(\gamma'^v, \zeta_j^v)\gamma'^v = 0.$$

Using the known solutions of Jacobi fields in M_c^n and $\mathbb{R}^m$, we obtain

$$\begin{cases} b_{ij}(r) = \delta_{ij} - a_{ij} r, & i \le m, \\ b_{ij}(r) = \delta_{ij} C_\tau(r) - a_{ij} S_\tau(r), & i > m, \end{cases} \tag{5.2}$$

where $S_\tau(r)$ and $C_\tau(r)$ are defined by

$$S_\tau(r) := \begin{cases} \dfrac{1}{\sqrt{c\tau^2}} \sin\left(\sqrt{c\tau^2}\, r\right), & c > 0, \\ \dfrac{1}{\sqrt{-c\tau^2}} \sinh\left(\sqrt{-c\tau^2}\, r\right), & c < 0, \end{cases} \qquad C_\tau(r) := \begin{cases} \cos\left(\sqrt{c\tau^2}\, r\right), & c > 0, \\ \cosh\left(\sqrt{-c\tau^2}\, r\right), & c < 0. \end{cases}$$

Moreover, these functions satisfy the first-order differential relations

$$S_\tau'(r) = C_\tau(r), \qquad C_\tau'(r) = -c\tau^2 S_\tau(r). \tag{5.3}$$

In fact, the matrix $B(r) = (b_{ij}(r))$ given in equation (5.2) can be written as the block matrix

$$B(r) = \left(\begin{array}{c|c} \delta_{ij} - a_{ij} r & -a_{ij} r \\ \hline -a_{ij} S_\tau(r) & \delta_{ij} C_\tau(r) - a_{ij} S_\tau(r) \end{array}\right). \tag{5.4}$$

By Jacobi field theory, $B(r)$ is nonsingular for all $r \in (-\delta, \delta)$, and the shape operator of Σ_r is given by

$$A_r = -B'(r)B(r)^{-1} \quad \text{(cf. [2, Theorem 10.2.1])}.$$

Hence, the mean curvature $H(r)$ is given by

$$H(r) = \operatorname{tr} A_r = -\operatorname{tr}\Big(B'(r)B(r)^{-1}\Big) = -\frac{d}{dr}(\det B(r))/\det B(r).$$

Defining $D(r) := \det B(r)$ and differentiating, we obtain

$$D'(r) + H(r)D(r) = 0,$$

that is,

$$D'(r) = -H(r)D(r).$$

By differentiating this equation repeatedly, for all $k \in \mathbb{N}$ we have

$$0 = D^{(k+1)}(r) + \phi_k(r)D(r), \tag{5.5}$$

where

$$\phi_k(r) = \phi_k\Big(H(r), H'(r), \dots, H^{(k)}(r)\Big).$$

Recalling the structure of the matrix $B(r)$ in (5.4), we observe that the highest power of r in the explicit expression for $D(r)$ is m. Hence, there exist coefficients $\alpha^q_{\ell,k}$ $(q = 0, \dots, m)$ such that

$$D^{(k)}(r) = \sum_{\ell=0}^{n-1}\sum_{q=0}^{m} \alpha^q_{\ell,k} r^q S^\ell_\tau(r) C^{n-1-\ell}_\tau(r), \tag{5.6}$$

where $D^{(k)}(r)$ denotes the k-th derivative of $D(r)$.

Substituting (5.6) into (5.5) and letting k vary from 1 to $(m+1)n-1$, we obtain

$$\alpha^0_{0,k+1} = -\phi_k(0). \tag{5.7}$$

Using (5.3) together with (5.6), we compute

$$\begin{aligned}
D^{(k+1)}(r) &= \sum_{\ell=0}^{n-1}\sum_{q=0}^{m} q\alpha^q_{\ell,k} r^{q-1} S^\ell_\tau(r) C^{n-1-\ell}_\tau(r) + \sum_{\ell=0}^{n-1}\sum_{q=0}^{m} \alpha^q_{\ell,k} r^q \ell S^{\ell-1}_\tau(r) C^{n-\ell}_\tau(r) \\
&\quad - \sum_{\ell=0}^{n-1}\sum_{q=0}^{m} \alpha^q_{\ell,k} r^q (n-1-\ell)c\tau^2 S^{\ell+1}_\tau(r) C^{n-2-\ell}_\tau(r) \\
&= \left(\sum_{q=0}^{m-1}\Big((q+1)\,\alpha^{q+1}_{0,k} + \alpha^q_{1,k}\Big) r^q + \alpha^m_{1,k} r^m\right) C^{n-1}_\tau(r) \\
&\quad + \sum_{\ell=1}^{n-2}\left(\sum_{q=0}^{m-1}\Big((q+1)\alpha^{q+1}_{\ell,k} + (\ell+1)\alpha^q_{\ell+1,k} - (n-\ell)c\tau^2\alpha^q_{\ell-1,k}\Big) r^q \right.\\
&\qquad\qquad \left. + \Big((\ell+1)\alpha^m_{\ell+1,k} - (n-\ell)c\tau^2\alpha^m_{\ell-1,k}\Big) r^m\right) S^\ell_\tau(r) C^{n-1-\ell}_\tau(r) \\
&\quad + \left(\sum_{q=0}^{m-1}((q+1)\alpha^{q+1}_{n-1,k} - c\tau^2\alpha^q_{n-2,k}) r^q - c\tau^2\alpha^m_{n-2,k} r^m\right) S^{n-1}_\tau(r).
\end{aligned}$$

Therefore, for $\ell = 0, \dots, n-1$ and $q = 0, \dots, m$, the coefficients satisfy

$$\alpha^q_{\ell,k+1} = (q+1)\alpha^{q+1}_{\ell,k} + (\ell+1)\alpha^q_{\ell+1,k} - (n-\ell)c\tau^2\alpha^q_{\ell-1,k}, \tag{5.8}$$

where we set $\alpha^{m+1}_{\ell,k} = 0$ for all $\ell = 0, \dots, n-1$ and $\alpha^q_{-1,k} = \alpha^q_{n,k} = 0$ for all $q = 0, \dots, m$.

From the recursive relation (5.8) among the coefficients $\alpha_{\ell,k}^q$, we may write

$$\alpha_{0,k+1}^0 = \sum_{\ell=0}^{n-1}\sum_{q=0}^{m} p_{k+1,\ell}^q\, \alpha_{\ell,0}^q, \tag{5.9}$$

where each coefficient $p_{k+1,\ell}^q$ depends only on the parameters q, k, ℓ, n, m, c, and τ.

Since $\alpha_{0,0}^0 = D(0) = 1$ and $\alpha_{0,k+1}^0$ coincides with $\phi_k(0)$ in equation (5.7), we conclude that the vector

$$\xi_0 = (\alpha_{1,0}^0, \dots, \alpha_{n-1,0}^0, \alpha_{0,0}^1, \dots, \alpha_{n-1,0}^{m-1}, \alpha_{0,0}^m, \dots, \alpha_{n-1,0}^m)^T \in \mathbb{R}^{(m+1)n-1}$$

satisfies a linear system of the form $M\xi = \nu$, according to (5.7), where

$$\nu = (-\phi_1(0) - p_{2,0}^0, \dots, -\phi_{(m+1)n-1}(0) - p_{(m+1)n,0}^0)^T \in \mathbb{R}^{(m+1)n-1}.$$

In the following, we shall see that the matrix M exhibits fundamentally different properties depending on whether n is odd or even. For odd n, we further denote by M^s the $((m+1)n-1)\times((m+1)n-1)$ matrix obtained from $\widetilde{M}^s$ in Proposition 5.5-(ii) by removing its first column. For any n, let M_ι and M_ι^s denote the matrices obtained by replacing the ι-th column of M and M^s, respectively, with the vector ν.

We establish key properties of M and M^s (for $s \geq (m+1)n$), in particular deriving a non-trivial algebraic expression in τ. Since the full proof is rather technical, it is deferred to the end of this section.

Proposition 5.1. *The matrices M (for even n) and M^s (for odd n and any $s \geq (m+1)n$) satisfy the following properties:*

(i) $\operatorname{rank} M = (m+1)n - 2$ *and* $\operatorname{rank} M^s = (m+1)n - 2$;

(ii) *There exists $\iota \in \{1, \dots, (m+1)n-1\}$ such that*

$$\det M_\iota = (-1)^{\frac{\gamma_0}{2}}\beta_0 c^{\frac{\gamma_0}{2}}\tau^{\gamma_0} - \sum_{i=1}^{(m+1)n-1}(-1)^{\frac{\gamma_i}{2}}\beta_i\phi_i(0)c^{\frac{\gamma_i}{2}}\tau^{\gamma_i},$$

where $\beta_0 \neq 0$, and $\beta_1, \dots, \beta_{(m+1)n-1}$ as well as $\gamma_0 > \dots > \gamma_{(m+1)n-1} > 0$ are integers;

(iii)

$$\det M_n^s = -(-1)^{\frac{\gamma_s}{2}}\beta_s\phi_s(0)c^{\frac{\gamma_s}{2}}\tau^{\gamma_s} - \sum_{i=1}^{(m+1)n-2}(-1)^{\frac{\gamma_i}{2}}\beta_i\phi_i(0)c^{\frac{\gamma_i}{2}}\tau^{\gamma_i},$$

where $\beta_s \neq 0$, $\beta_1, \dots, \beta_{(m+1)n-2}$ as well as $\gamma_1 > \dots > \gamma_{(m+1)n-2} > \gamma_s > 0$ are all integers.

Proof of Theorem 1.1.

We will primarily apply the non-trivial algebraic expression in τ implied by M and M^s as established in Proposition 5.1, and prove Theorem 1.1 proceeding case by case.

Case 1: $n \geq 2$ and n is even. By Proposition 5.1-(i), $\det M = 0$. Since ξ_0 satisfies $M\xi = \nu$, it follows that $\det M_i = 0$ for all $i = 1, \dots, (m+1)n-1$. Moreover, by Proposition 5.1-(ii), there exists an index $\iota \in \{1, \dots, (m+1)n-1\}$ such that

$$\det M_\iota = (-1)^{\frac{\gamma_0}{2}}\beta_0 c^{\frac{\gamma_0}{2}}\tau^{\gamma_0} - \sum_{i=1}^{(m+1)n-1}(-1)^{\frac{\gamma_i}{2}}\beta_i\phi_i(0)c^{\frac{\gamma_i}{2}}\tau^{\gamma_i} = 0.$$

This yields a nontrivial algebraic equation in τ, and hence τ is constant.

Case 2: $n \geq 2$ and n is odd. If there exists $s_0 \geq (m+1)n$ with $\phi_{s_0}(0) \neq 0$, then, analogously to Case 1, Proposition 5.1-(i) and (iii) imply that

$$\det M_n^{s_0} = 0$$

is a nontrivial algebraic equation in τ, so τ is constant.

Otherwise, if $\phi_s(0) = 0$ for all $s \geq (m+1)n$, then equation (5.5) shows that $D(r)$ is polynomial near $r = 0$. However, from equation (5.4), $D(r) = \det B(r)$ cannot be polynomial near $r = 0$, so this case is excluded.

□

5.1. **Proof of Proposition 5.1.** We first derive the recurrence relation in k for the coefficients $p^q_{k+1,\ell}$ in equation (5.9). For $n \geq 4$, combining equations (5.8) and (5.9), we obtain the following explicit computation:

$$\begin{aligned}
\alpha^0_{0,k+1} &= \sum_{\ell=0}^{n-1}\sum_{q=0}^{m} p^q_{k,\ell}\alpha^q_{\ell,1} \\
&= \sum_{q=0}^{m} p^q_{k,0}\alpha^q_{0,1} + \sum_{\ell=1}^{n-2}\sum_{q=0}^{m} p^q_{k,\ell}\alpha^q_{\ell,1} + \sum_{q=0}^{m} p^q_{k,n-1}\alpha^q_{n-1,1} \\
&= \sum_{q=1}^{m} p^{q-1}_{k,0} q\alpha^q_{0,0} + \sum_{q=0}^{m-1} p^q_{k,0}\alpha^q_{1,0} + p^m_{k,0}\alpha^m_{1,0} \\
&\quad + \sum_{\ell=1}^{n-2}\sum_{q=1}^{m} p^{q-1}_{k,\ell} q\alpha^q_{\ell,0} + \sum_{\ell=2}^{n-1}\sum_{q=0}^{m-1} p^q_{k,\ell-1}\ell\alpha^q_{\ell,0} - \sum_{\ell=0}^{n-3}\sum_{q=0}^{m-1} p^q_{k,\ell+1}(n-1-\ell)c\tau^2\alpha^q_{\ell,0} \\
&\quad + \sum_{\ell=2}^{n-1} p^m_{k,\ell-1}\ell\alpha^m_{\ell,0} - \sum_{\ell=0}^{n-3} p^m_{k,\ell+1}(n-1-\ell)c\tau^2\alpha^m_{\ell,0} \\
&\quad + \sum_{q=1}^{m} p^{q-1}_{k,n-1} q\alpha^q_{n-1,0} - \sum_{q=0}^{m-1} p^q_{k,n-1}c\tau^2\alpha^q_{n-2,0} - p^m_{k,n-1}c\tau^2\alpha^m_{n-2,0} \\
&= \sum_{\ell=0}^{n-1}\sum_{q=0}^{m}\left(p^{q-1}_{k,\ell}q + p^q_{k,\ell-1}\ell - p^q_{k,\ell+1}(n-1-\ell)c\tau^2\right)\alpha^q_{\ell,0}.
\end{aligned}$$

For convenience, setting $p^q_{k,\ell} = 0$ if $q = -1$ or $\ell = -1, n$, we have

$$p^q_{k+1,\ell} = q\,p^{q-1}_{k,\ell} + \ell\,p^q_{k,\ell-1} - (n-1-\ell)c\tau^2 p^q_{k,\ell+1}, \tag{5.10}$$

for any $q = 0, \ldots, m$ and $\ell = 0, \ldots, n-1$.

Remark 5.2. *For $n = 2$ and $n = 3$, the computations are entirely analogous and remain relatively straightforward. Although we omit the explicit calculations for brevity, these low-dimensional cases reproduce the formula in equation (5.10) exactly. This confirms that equation (5.10) holds for all integers $n \geq 2$.*

More specifically, we have $p^0_{0,0} = 1$ and $p^q_{0,\ell} = 0$ for all other (ℓ, q), since $\alpha^0_{0,0} = 1$ and equation (5.9). Taking $k = 0$ in (5.10), we obtain

$$p^1_{1,0} = p^0_{1,1} = 1, \quad p^q_{1,\ell} = 0 \quad \text{for } (\ell, q) \neq (0,1), (1,0).$$

Similarly, when $k = 1$, we find that

$$p^q_{2,\ell} = \begin{cases} -(n-1)c\tau^2, & \text{if } (\ell, q) = (0,0), \\ 2, & \text{if } (\ell, q) = (0,2), (1,1), (2,0), \\ 0, & \text{otherwise.} \end{cases} \tag{5.11}$$

By applying the recurrence relation (5.10) and mathematical induction, we obtain the following basic characterization of $p^q_{k,\ell}$:

Proposition 5.3. *When $n \geq 2$ and for any $k \geq 2$, $q = 0, \dots, m$, $\ell = 0, \dots, n-1$, the identity $p^q_{k,\ell} = \sigma^q_{k,\ell}(n)c^s\tau^{2s}$ holds with $s = \frac{1}{2}(k - q - \ell)$. Furthermore, the following assertions also hold:*

(i) *$\sigma^q_{k,\ell}(n) = 0$, for $s \notin \mathbb{Z}$ or $s < 0$.*

(ii) *$\sigma^q_{k,\ell}(n) = k!$, for $s = 0$.*

(iii) *$\sigma^q_{k,\ell}(n)$ is a polynomial of degree $\deg \sigma^q_{k,\ell}(n) \geq s$ with $(-1)^s$ as its leading coefficient sign for $s \in \mathbb{Z}_+$.*

Proof. (i) If $s \notin \mathbb{Z}$, then $k - q - \ell$ is odd. As discussed above, $p^0_{0,0} = 1$ and $p^q_{0,\ell} = 0$ for all $\ell = 1, \dots, n-1$ and $q = 0, \dots, m$. Since the parity of $k - q - \ell$ is preserved in equation (5.10), it follows that $p^q_{k,\ell} = 0$ for all $s \notin \mathbb{Z}$.

If $s < 0$, then $q + \ell > k$. By induction, we will show that $p^q_{k,\ell} = 0$ also holds in this case.

For $k = 2$, by equation (5.11), we have

$$p^q_{2,\ell} = 0, \quad \text{for all } q, \ell \text{ with } q + \ell > 2.$$

Now assume that for some $k \geq 2$, $p^q_{k,\ell} = 0$ for all (q, ℓ) satisfying $q + \ell > k$. Then for any q, ℓ with $q + \ell > k + 1$, equation (5.10) gives

$$p^q_{k+1,\ell} = q\, p^{q-1}_{k,\ell} + \ell\, p^q_{k,\ell-1} - (n - 1 - \ell)c\tau^2 p^q_{k,\ell+1}.$$

By the induction hypothesis, $p^{q-1}_{k,\ell} = p^q_{k,\ell-1} = p^q_{k,\ell+1} = 0$ since $(q-1)+\ell > k$, $q+(\ell-1) > k$, and $q + (\ell + 1) > k$. Therefore, $p^q_{k+1,\ell} = 0$, as required.

(ii) It suffices to show that $p^q_{k,k-q} = k!$ for all such k, q. When $k = 2$, equation (5.11) gives $p^0_{2,2} = 2$.

Suppose that for some integer $k \geq 2$, the identity $p^q_{k,k-q} = k!$ holds for all q. Then, for any q satisfying $k + 1 \geq 2 + q$, equation (5.10) implies

$$\begin{aligned} p^q_{k+1,k+1-q} &= q\, p^{q-1}_{k,k+1-q} + (k + 1 - q)\, p^q_{k,k-q} - (n - 1 - (k + 1 - q))c\tau^2 p^q_{k,k+2-q} \\ &= (k + 1)!, \end{aligned}$$

where the last equality uses $p^{q-1}_{k,k+1-q} = p^q_{k,k-q} = k!$ from the induction hypothesis and $p^q_{k,k+2-q} = 0$ from (i). Hence, the result follows by induction.

(iii) Equivalently, it suffices to show that for any $k \geq 2$,

$$\deg \sigma^q_{k,\ell}(n) \geq s, \qquad \text{where } \ell = k - q - 2s,$$

for all integers s and q satisfying $0 < s \leq \frac{1}{2}(k - q)$.

To prove this, we start with $k = 2$ and proceed by induction. From equation (5.11), the only term satisfying the condition is $p^0_{2,0} = -(n-1)c\tau^2$. Hence $\deg \sigma^0_{2,0}(n) = s = 1$, verifying the claim for $k = 2$.

Assume now that for some $k \geq 2$, one has $\deg \sigma^q_{k,\,k-q-2s}(n) \geq s$ for all integers q, s satisfying $0 < s \leq \frac{1}{2}(k-q)$, and that the leading coefficient has sign $(-1)^s$. For $k+1$, take any such q, s with $0 < s \leq \frac{1}{2}(k+1-q)$ and set $\ell = k+1-q-2s$. By equation (5.10), we obtain

$$\begin{aligned} p^q_{k+1,\ell} &= q\, p^{q-1}_{k,\ell} + \ell\, p^q_{k,\ell-1} - (n-1-\ell)c\tau^2 p^q_{k,\ell+1} \\ &= q\, \sigma^{q-1}_{k,\ell}(n)c^s\tau^{2s} + \ell\, \sigma^q_{k,\ell-1}(n)c^s\tau^{2s} - (n-1-\ell)\sigma^q_{k,\ell+1}(n)c^s\tau^{2s} \\ &= \sigma^q_{k+1,\ell}(n)c^s\tau^{2s}, \end{aligned}$$

where

$$\sigma^q_{k+1,\ell}(n) = q\, \sigma^{q-1}_{k,\ell}(n) + \ell\, \sigma^q_{k,\ell-1}(n) - (n-1-\ell)\sigma^q_{k,\ell+1}(n).$$

By the induction hypothesis, we have $\deg \sigma^{q-1}_{k,\ell}(n) = \deg \sigma^q_{k,\ell-1} = s$ and $\deg \sigma^q_{k,\ell+1} = s-1$, with respective leading coefficient signs $(-1)^s, (-1)^s$, and $(-1)^{s-1}$. It then follows that $\deg \sigma^q_{k+1,\ell}(n) \geq s$ and its leading coefficient has sign $(-1)^s$, which completes the proof.

□

To study the rank properties of M and M^s $(s \geq (m+1)n)$, we adopt a row-wise perspective. Define the matrix $\widetilde{M} = [-\nu_\tau,\, M]$ where $\nu_\tau = \nu - \nu_\phi$ and

$$\nu_\phi = (-\phi_1(0), \ldots, -\phi_{(m+1)n-1}(0))^T.$$

Let M^τ_ι (resp., M^ϕ_ι) denote the matrix obtained from M by replacing its ι-th column with ν_τ (resp., ν_ϕ). Under this setting, each row of the matrix $\widetilde{M}$ is a row vector of the form

$$\widetilde{L}_{k-1} = \left(p^0_{k,0}, \ldots, p^0_{k,n-1},\, p^1_{k,0}, \ldots, p^1_{k,n-1},\, \ldots,\, p^m_{k,0}, \ldots, p^m_{k,n-1}\right), \quad k \geq 2,$$

where each segment $p^q_{k,0}, \ldots, p^q_{k,n-1}$ corresponds to $q = 0, \ldots, m$.

Define $e_1 \in \mathbb{R}^n$ by

$$e_1 = (p^0_{0,0}, p^0_{0,1}, \ldots, p^0_{0,n-1}) = (1, 0, \ldots, 0).$$

Next, we define an $(m+1)n \times (m+1)n$ matrix Q as follows:

$$Q = \begin{pmatrix} K & I & & & & & \\ & K & 2I & & & & \\ & & K & 3I & & & \\ & & & \ddots & \ddots & & \\ & & & & K & (m-1)I & \\ & & & & & K & mI \\ & & & & & & K \end{pmatrix}, \tag{5.12}$$

where I is the $n \times n$ identity matrix and K is the $n \times n$ τ-Kac matrix

$$K = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -(n-1)c\tau^2 & 0 & 2 & \cdots & 0 & 0 & 0 \\ 0 & -(n-2)c\tau^2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 & 0 \\ 0 & 0 & 0 & \cdots & -2c\tau^2 & 0 & n-1 \\ 0 & 0 & 0 & \cdots & 0 & -c\tau^2 & 0 \end{pmatrix}.$$

Then, for $k \geq 2$, using the recurrence equation (5.10), the rows of $\widetilde{M}$ satisfy

$$\widetilde{L}_k = \widetilde{L}_{k-1} Q = (e_1, 0, \ldots, 0) Q^{k+1}.$$

Regarding the properties of the τ-Kac matrix, we recall the following lemma established in [11, 14].

Lemma 5.4. ([11, 14]) *The τ-Kac matrix K of order n has the following propertues:*

(i) *It has n simple eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$, which are*

$$\lambda_\ell = (n - 1 - 2\ell)\sqrt{-c}\tau, \quad \ell \in \{0, \ldots, n-1\}.$$

In particular λ_ℓ is real if $c < 0$, and purely imaginarg if $c > 0$;

(ii) *Its rank is n, if n is even, and $n-1$ if n is odd. In particular, K is nonsignular if and only if n is even;*

(iii) *The coordinates of $e_1 \in \mathbb{R}^n$ with respect to the basis of its eigenvectors are all different from zero.*

We now relate Q to the τ-Kac matrix K, in particular its eigenvectors. Direct computation gives

$$\det Q = (\det K)^{m+1},$$

and thus by Lemma 5.4-(ii), Q is nonsingular if and only if n is even.

Let $\{x_0, \ldots, x_{n-1}\} \subset \mathbb{R}^n$ be the eigenvectors of K. For $\ell = 0, \ldots, n-1$, we define the following vector in $\mathbb{R}^{(m+1)n}$:

$$x_{0,\ell} = (x_\ell, 0, \ldots, 0), \quad x_{1,\ell} = (0, x_\ell, 0, \ldots, 0), \quad \ldots, \quad x_{m,\ell} = (0, \ldots, 0, x_\ell).$$

Then, we have

$$x_{k,\ell} Q = \lambda_\ell x_{k,\ell} + x_{k+1,\ell}, \quad k = 0, \ldots, m-1, \qquad x_{m,\ell} Q = \lambda_\ell x_{m,\ell}.$$

More generally, for any integer $k \geq 0$ and $0 \leq i \leq m$, by induction,

$$x_{i,\ell} Q^k = \sum_{t=0}^{\min\{k, m-i\}} \binom{k}{t} \lambda_\ell^{k-t} x_{i+t,\ell}. \tag{5.13}$$

Proposition 5.5. *Let $\widetilde{e}_1 = (e_1, 0, \ldots, 0) \in \mathbb{R}^{(m+1)n}$ with $n \geq 2$. Then*

(i) *If n is even, for any positive integer s, the set*

$$\{\widetilde{e}_1 Q^i \mid i = s, \ldots, s + (m+1)n - 1\}$$

is linearly independent.

(ii) *If n is odd, for any integer $s \geq (m+1)n$, define the ordered sets*

$$\Lambda = \{\widetilde{e}_1 Q^i \mid i = 2, \dots, (m+1)n-1\}, \qquad \Lambda_s = \Lambda \cup \{\widetilde{e}_1 Q^s\}.$$

Let $\widetilde{M}^s$ be the matrix with rows given by the vectors in Λ_s, and denote its columns by $C_1, \dots, C_{(m+1)n}$. Then the following hold:

(a) *Λ is linearly independent, whereas Λ_s is linearly dependent.*

(b) *For $q = 0, 1$, the column C_{qn+1} lies in the span of the columns $\{C_{qn+2i+1} \mid i = 1, \dots, (n-1)/2\}$.*

Proof. (i) When n is even, the previous calculations show that Q is invertible, so it suffices to consider $s = 0$.

Consider the vector equation

$$\sum_{k=0}^{(m+1)n-1} \mu_k \widetilde{e}_1 Q^k = 0 \tag{5.14}$$

in the variables $\mu_0, \dots, \mu_{(m+1)n-1}$.

Without loss of generality, by Lemma 5.4-(iii), we may write

$$\widetilde{e}_1 = \sum_{\ell=0}^{n-1} a_\ell x_{0,\ell},$$

where $a_\ell \neq 0$ for all $\ell = 0, \dots, n-1$.

From equation (5.13), we obtain

$$\sum_{\ell=0}^{n-1} \sum_{k=0}^{(m+1)n-1} \sum_{t=0}^{\min\{k,m\}} \binom{k}{t} \lambda_\ell^{k-t} \mu_k a_\ell x_{t,\ell} = 0.$$

Hence, the system (5.14) is equivalent to the linear system

$$\sum_{k=0}^{(m+1)n-1} \binom{k}{t} \lambda_\ell^{k-t} \mu_k = 0, \quad t = 0, \dots, m, \ \ell = 0, \dots, n-1. \tag{5.15}$$

The coefficient matrix Ξ of (5.15) is a generalized Vandermonde matrix with

$$\det \Xi = \prod_{i<j} (\lambda_j - \lambda_i)^{(m+1)^2}.$$

By Lemma 5.4, the eigenvalues λ_ℓ are distinct, so $\det \Xi \neq 0$ and Ξ is invertible. Hence, $\mu_k = 0$ for all $k = 0, \dots, (m+1)n-1$, completing the proof of (i).

(ii)-(a) Similar to (i), for any $s \geq (m+1)n$, we consider the following vector equation in the variables $\mu_2, \dots, \mu_{(m+1)n-1}, \mu_s$:

$$\sum_{k=2}^{(m+1)n-1} \mu_k \widetilde{e}_1 Q^k + \mu_s \widetilde{e}_1 Q^s = 0,$$

which is equivalent to the linear system

$$\sum_{k=2}^{(m+1)n-1} \binom{k}{t} \lambda_\ell^{k-t} \mu_k + \binom{s}{t} \lambda_\ell^{s-t} \mu_s = 0, \quad t = 0, \dots, m, \ \ell = 0, \dots, n-1. \tag{5.16}$$

Since n is odd, Lemma 5.4-(i) gives $\lambda_{(n-1)/2} = 0$. Thus, (5.16) is a linear system of $(m+1)n-2$ equations in $(m+1)n-1$ unknowns. Its coefficient matrix Ξ has block form

$$\Xi = \begin{pmatrix} \Xi_0 \\ \Xi_1 \\ \vdots \\ \Xi_{\frac{n-3}{2}} \\ \Xi_{\frac{n-1}{2}} \\ \Xi_{\frac{n+1}{2}} \\ \vdots \\ \Xi_{n-1} \end{pmatrix},$$

where the block Ξ_ℓ is the $(m+1)\times((m+1)n-1)$ matrix

$$\Xi_\ell = \begin{pmatrix} \lambda_\ell^2 & \lambda_\ell^3 & \lambda_\ell^4 & \cdots & \lambda_\ell^{(m+1)n-1} & \lambda_\ell^s \\ 2\lambda_\ell & 3\lambda_\ell^2 & 4\lambda_\ell^3 & \cdots & ((m+1)n-1)\lambda_\ell^{(m+1)n-2} & s\lambda_\ell^{s-1} \\ 1 & 3\lambda_\ell & 6\lambda_\ell^2 & \cdots & \binom{(m+1)n-1}{2}\lambda_\ell^{(m+1)n-3} & \binom{s}{2}\lambda_\ell^{s-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{(m+1)n-1}{m}\lambda_\ell^{(m+1)(n-1)} & \binom{s}{m}\lambda_\ell^{s-m} \end{pmatrix}.$$

In particular,

$$\Xi_{\frac{n-1}{2}} = \begin{pmatrix} \mathrm{O} & \mathrm{O} \\ I_{m-1} & \mathrm{O} \end{pmatrix}. \tag{5.17}$$

Hence, $\operatorname{rank}\Xi \le (m+1)n-2$, implying that the set Λ_s is linearly dependent.

To prove that Λ is independent, remove the last column of Ξ (corresponding to μ_s) to form $\tilde{\Xi}$. Expanding the block $\tilde{\Xi}_{\frac{n-1}{2}}$ in equation (5.17) and applying generalized Vandermonde determinant properties, we obtain

$$\det\tilde{\Xi} = \prod_{\substack{i<j \\ i,j\neq(n-1)/2}} (\lambda_j-\lambda_i)^{(m+1)^2} \cdot \prod_{i\neq(n-1)/2} \lambda_i^{(m+1)^2} \neq 0.$$

Hence, $\tilde{\Xi}$ is nonsingular, and Λ is linearly independent.

(ii)-(b) By equation (5.12) and induction, we have

$$Q^j = \begin{pmatrix} K^j & \binom{j}{1}1^{\overline{1}}K^{j-1} & \binom{j}{2}1^{\overline{2}}K^{j-2} & \cdots & \binom{j}{m}1^{\overline{m}}K^{j-m} \\ 0 & K^j & \binom{j}{1}2^{\overline{1}}K^{j-1} & \cdots & \binom{j}{m-1}2^{\overline{m-1}}K^{j-m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{j}{1}m^{\overline{1}}K^{j-1} \\ 0 & 0 & 0 & \cdots & K^j \end{pmatrix}, \tag{5.18}$$

or equivalently,

$$Q^j[p,q] = \begin{cases} \binom{j}{d} p^{\overline{d}} K^{j-d}, & q = p+d,\ 1 \le d \le m, \\ K^j, & q = p, \\ 0, & \text{otherwise}, \end{cases}$$

where $p^{\overline{d}} = p(p+1)\cdots(p+d-1)$ is the rising factorial and $Q^j[p,q]$ is the element in the p-th row and q-th column of the matrix Q^j.

Similar to above argument in (ii)-(a), the last row will be immaterial. Without loss of generality, assume $s = (m+1)n$. By equation (5.18), we note that $C_{qn+1}, C_{qn+2}, \ldots, C_{(q+1)n}$ are the columns of the matrix whose rows are

$$e_1 \binom{2}{q} K^{2-q},\ e_1 \binom{3}{q} K^{3-q},\ \ldots,\ e_1 \binom{(m+1)n}{q} K^{(m+1)n-q},$$

for $q = 0, 1$.

We claim that the set $\{C_{qn+2i+1} \mid i = 0, \ldots, (n-1)/2\}$ spans a space of dimension $(n-1)/2$. Consider the vector equation in $\bar{n} := \left\lfloor \frac{(m+1)n-q}{2} \right\rfloor$ variables $\mu_1, \ldots, \mu_{\bar{n}}$:

$$\sum_{j=1}^{\bar{n}} \mu_j \binom{2j+q}{q} e_1 K^{2j} = 0,$$

which is equivalent to the linear system

$$\sum_{j=1}^{\bar{n}} \binom{2j+q}{q} \lambda_\ell^{2j} \mu_j = 0, \quad \ell = 0, \ldots, n-1. \tag{5.19}$$

The coefficient matrix of the linear system (5.19) is

$$\Xi = \begin{pmatrix} \binom{2+q}{q}\lambda_0^2 & \binom{4+q}{q}\lambda_0^4 & \cdots & \binom{2\bar{n}+q}{q}\lambda_0^{2\bar{n}} \\ \binom{2+q}{q}\lambda_1^2 & \binom{4+q}{q}\lambda_1^4 & \cdots & \binom{2\bar{n}+q}{q}\lambda_1^{2\bar{n}} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{2+q}{q}\lambda_{n-1}^2 & \binom{4+q}{q}\lambda_{n-1}^4 & \cdots & \binom{2\bar{n}+q}{q}\lambda_{n-1}^{2\bar{n}} \end{pmatrix}.$$

By Lemma 5.4-(i), $\lambda_{(n-1)/2} = 0$, and the nonzero eigenvalues occur in pairs $\pm\lambda$, hence $\operatorname{rank}\Xi \le (n-1)/2$. Taking the first $(n-1)/2$ columns and rows corresponding to distinct eigenvalues gives a submatrix $\tilde{\Xi}$ with

$$\det \tilde{\Xi} = \prod_{i<j} (\lambda_j^2 - \lambda_i^2) \cdot \prod_{\ell=0}^{(n-3)/2} \lambda_\ell^2 \neq 0.$$

Therefore, we obtain that $\operatorname{rank}\Xi = \frac{n-1}{2}$ and the claim is proved.

Finally, by Proposition 5.3-(i) and (ii), the submatrix consisting of the first $\frac{n-1}{2}$ nonzero rows of the matrix formed by the column vectors $\{C_{qn+2i+1} \mid i = 1, \ldots, \frac{n-1}{2}\}$ is a lower triangular matrix, whose diagonal entries are $(2+q)!, (4+q)!, \ldots, (q+n-1)!$. Therefore, the vectors $\{C_{qn+2i+1} \mid i = 1, \ldots, \frac{n-1}{2}\}$ are linearly independent, and C_{qn+1} lies in their span. This completes the proof.

□

Proof of Proposition 5.1. (i) When n is even. By Proposition 5.3-(i), we observe that all odd rows of matrix M form a matrix with $\frac{(m+1)n}{2}$ rows but only $\frac{(m+1)n}{2}-1$ nonzero columns. Hence, the rank of the odd rows is at most $\frac{(m+1)n}{2}-1$, implying

$$\operatorname{rank} M \leq (m+1)n-2.$$

On the other hand, applying Proposition 5.5-(i) with $s=2$, the augmented matrix $\widetilde{M}$ has rank $(m+1)n-1$, which shows

$$\operatorname{rank} M \geq (m+1)n-2.$$

Combining the bounds, we conclude that $\operatorname{rank} M = (m+1)n-2$.

When n is odd. Consider $s=(m+1)n$ in Proposition 5.5-(ii). The augmented matrix $\widetilde{M}^s$ then has rank $(m+1)n-2$. Moreover, by Proposition 5.5-(ii)-(b) with $q=0$, the first column C_1 lies in the span of $C_3, C_5, \ldots, C_n$. Therefore,

$$\operatorname{rank} M^s = \operatorname{rank} \widetilde{M}^s = (m+1)n-2.$$

(ii) By Proposition 5.5-(i), the augmented matrix $\widetilde{M}$ has rank $(m+1)n-1$. Hence, there exists an index $\iota \in \{1, \ldots, (m+1)n-1\}$ such that

$$\det M_\iota^\tau \neq 0.$$

By Proposition 5.3-(iii), there exist a nonzero integer β_0 and a positive integer $\gamma_0 > 0$ such that

$$\det M_\iota^\tau = (-1)^{\gamma_0/2}\beta_0\, c^{\gamma_0/2}\tau^{\gamma_0}.$$

Next, performing a Laplace expansion along the ι-th column of $\det M_\iota^\phi$, we obtain the form described in (ii), with constant coefficients $\{\beta_k\}_{k=1}^{(m+1)n-1}$ and exponents $\{\gamma_k\}_{k=1}^{(m+1)n-1}$. Meanwhile, Proposition 5.3 implies that all γ_i are even, each β_i is an integer, and that the sequence of γ_i is strictly increasing in i.

Finally, the positivity of $\gamma_{(m+1)n-1}$ follows from the estimate:

$$\begin{aligned}
\gamma_{(m+1)n-1} &\geq \sum_{i=1}^{(m+1)n-2}(i-1) + \sum_{q=0}^{m}\sum_{j=1}^{n-1}\Big(q(n-1)-j+2\Big) \\
&= \frac{1}{2}\Big(m^2(2n^2-2n+1)+m(2n^2-2n-3)\Big)+1>0.
\end{aligned}$$

(iii) Define the matrices M_ι^s, $M_\iota^{s,\tau}$, and $M_\iota^{s,\phi}$ analogously to M_ι, M_ι^τ, and M_ι^ϕ, respectively. It follows from Proposition 5.5-(ii) that

$$\det M_\iota^{s,\tau} = 0, \quad \text{for all } \iota = 1, \ldots, (m+1)n-1.$$

In particular, for $\iota = n$, this yields

$$\det M_n^{s,\tau} = 0.$$

Substituting into the identity

$$\det M_n^s = \det M_n^{s,\tau} + \det M_n^{s,\phi}$$

immediately gives

$$\det M_n^s = \det M_n^{s,\phi}.$$

Applying the same methodology as in the proof of part (ii) then produces the asserted expression for $\det M_n$.

Finally, we verify that $\beta_s \neq 0$. Indeed, β_s is the determinant of the submatrix obtained by removing the last row and the n-th column from M^s. Proposition 5.5-(ii) ensures that this submatrix is nonsingular, and hence $\beta_s \neq 0$.

□

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