Self-dualities and Galois symmetries in Feynman integrals

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Abstract

It is well-known that all Feynman integrals within a given family can be expressed as a finite linear combination of master integrals. The master integrals naturally group into sectors. Starting from two loops, there can exist sectors made up of more than one master integral. In this paper we show that such sectors may have additional symmetries. First of all, self-duality, which was first observed in Feynman integrals related to Calabi–Yau geometries, often carries over to non-Calabi–Yau Feynman integrals. Secondly, we show that in addition there can exist Galois symmetries relating integrals. In the simplest case of two master integrals within a sector, whose definition involves a square root r, we may choose a basis (I_1,I_2) such that I_2 is obtained from I_1 by the substitution $r \to -r$. This pattern also persists in sectors, which a priori are not related to any square root with dependence on the kinematic variables. We show in several examples that in such cases a suitable redefinition of the integrals introduces constant square roots like $\sqrt{3}$. The new master integrals are then again related by a Galois symmetry, for example the substitution $\sqrt{3} \to -\sqrt{3}$. To handle the case where the argument of a square root would be a perfect square we introduce a limit Galois symmetry. Both self-duality and Galois symmetries constrain the differential equation.

1 Introduction

Integration-by-parts and differential equations are among the most popular tools to compute Feynman integrals. Integration-by-parts allows us to express any integral from a family of Feynman integrals as a linear combination of master integrals [1, 2]. The set of master integrals is finite [3]. The method of differential equations allows us to write down a differential equation for the master integrals with respect to the kinematic variables [4–7]. In order to solve the differential equation one tries to find a transformation to an ε -factorised form [8]

$$dI = \varepsilon AI, \tag{1}$$

where I denotes the vector of master integrals, the symbol d is the differential with respect to all kinematic variables, ε denotes the dimensional regularisation parameter, and A is a square matrix with dimensions equal to the number of master integrals. The entries of A are differential one-forms, depending on the kinematic variables, but independent of the dimensional regularisation parameter ε . A differential equation in ε -factorised form can be solved systematically order-by-order in ε in terms of iterated integrals [9].

In this paper we investigate the structure of the matrix A in more detail. In particular we show that it is often possible to redefine the master integrals, such that the ε -factorised form is maintained, but additional symmetries can be realised. This is achieved by a transformation

$$J = UI, (2)$$

where U is a $GL(n,\mathbb{C})$ -matrix, independent of the dimensional regularisation parameter ε and the kinematic variables. The number of master integrals is denoted by n. Such transformations preserve the ε -factorised form, as required.

Master integrals may be grouped into sectors. If we order the master integrals such that the first integral corresponds to the simplest and the last to the most complicated one, then the matrix A has a lower block-triangular structure. The size of the blocks on the diagonal is given by the number of master integrals in the corresponding sector. While at one loop, each sector has just one master integral, starting from two loops there may be sectors with more than one master integral. In this paper we are interested in sectors with more than one master integral. We show that it is often possible to redefine the integrals such that the master integrals within one sector exhibit a self-duality symmetry. Self-duality is the statement that the block on the diagonal is reflection-symmetric with respect to the anti-diagonal. This type of symmetry was first observed in Feynman integrals related to Calabi–Yau geometries [10,11]. Extending the analysis of ref. [12], we show that self-duality often extends to non-Calabi-Yau Feynman integrals. In addition there can exist Galois symmetries, such that master integrals within the same sector are related by the action of a Galois group. In the simplest case such an action amounts to flipping the sign of a specific square root r appearing in the integrals, i.e. $r \to -r$. Galois symmetries have been used in the past to group the letters of the symbol alphabet into even and odd letters, see for example refs. [13–15]. In this paper we show that Galois symmetries often extend from symmetry properties of individual letters to relations between master integrals.

Both types of symmetries, self-duality and Galois symmetries, induce relations among the entries of the differential equation matrix A. To give an example, consider a sector with two master integrals I_1, I_2 , and assume that we may choose the two master integrals such that self-duality and Galois symmetry are manifest. The block on the diagonal of the matrix A (corresponding to the maximal cut of this sector)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{3}$$

has then the self-duality symmetry

$$a_{11} = a_{22}.$$
 (4)

The Galois symmetry relates the two master integrals such that $I_2 = \sigma(I_1)$, where σ is an element of the Galois group. This induces on the matrix A the relations

$$a_{22} = \sigma(a_{11}), \quad a_{21} = \sigma(a_{12}).$$
 (5)

The combination of Galois symmetry and self-duality gives us therefore

$$a_{22} = a_{11}, \quad a_{21} = \sigma(a_{12}), \quad a_{11} = \sigma(a_{11}).$$
 (6)

We see that we only need to know two entries of the block, the other two entries follow from symmetries.

Galois symmetries are expected if the definition of one master integral in the sector involves a square root r. In the simplest case of two master integrals within one sector, whose definition involves a square root r, we may choose a basis (I_1,I_2) such that I_2 is obtained from I_1 by the substitution $r \to -r$. However, we can go further. We point out that this pattern may also appear in sectors that a priori are not related to any square root with dependence on the kinematic variables. We show that in these cases constant square roots such as $\sqrt{3}$ can be introduced, such that after a suitable redefinition the master integrals are related by the substitution $\sqrt{3} \to -\sqrt{3}$.

There is no reason to expect self-duality to extend to the full differential equation beyond a single sector: If we order the master integrals by sectors and the sectors from the simplest to the most complicated, self-duality would relate the most complicated sector to the simplest one, which is obviously impossible. However, we show in several examples that Galois symmetry does extend to the full matrix A. Consider a second sector consisting of two master integrals I_3 , I_4 , and let us assume that they have been chosen such that $I_4 = \sigma'(I_3)$, where σ' is another element of the Galois group not identical to the first one. Let us further assume that σ' acts trivially on I_1 and I_2 , and that σ acts trivially on I_3 and I_4 . The matrix A is thus a (4×4) -matrix. The lower-left non-diagonal block

$$\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \tag{7}$$

has then the symmetries

$$a_{32} = \sigma(a_{31}), \quad a_{41} = \sigma'(a_{31}), \quad a_{42} = \sigma'(\sigma(a_{31})).$$
 (8)

In this case we see that three entries out of four of the non-diagonal block may be obtained from Galois symmetries. Our strategy is always to impose self-duality first, and in a second step to realise Galois symmetries. Although we were able to make self-duality manifest in all examples we checked, we have no proof that this is possible in general. While sectors with two or more master integrals often satisfy in addition to self-duality a Galois symmetry, we are aware of integrals where this is not the case. The most prominent examples are the massless planar double-box and the equal-mass sunrise integral. In the case of the massless planar double-box integral we still have a "limit Galois symmetry". We will explain this concept in detail in the main part of the paper. In the case of the equal-mass sunrise integral we do not expect Galois symmetries, as Galois symmetries cannot relate quantities of different modular weight.

The transformation U, which realises self-duality and Galois symmetry is not necessarily unique. We provide a simple example with two master integrals in one sector and two square roots r_1 and r_2 related to this sector. We may either choose a basis such that $J_2 = \sigma_1(J_1)$ (where σ_1 corresponds to $r_1 \to -r_1$) or a basis $J_2' = \sigma_2(J_1')$ (where σ_2 corresponds to $r_2 \to -r_2$). In a second example with three master integrals in one sector we show that there is even a one-parameter family of possible transformations, which realise self-duality and Galois symmetry.

This paper is organised as follows: In the following section we review the concept of master integrals, self-duality and the basics of Galois theory. In section 3 we show in several examples that it is possible to realise self-duality and Galois symmetries. We first focus on examples, where the differential one-forms are dlog-forms with algebraic arguments. We are in particular interested in the case, where square roots appear in the arguments of the dlog-forms. For our study it is irrelevant, if the square roots can be rationalised simultaneously or not. We present examples for both cases. In section 4 we comment on Feynman integrals, whose differential equations involve differential one-forms beyond dlog-forms with algebraic arguments. This includes the elliptic case. Finally, section 5 contains our conclusions.

2 Set-up

2.1 Master integrals

We consider Feynman integrals, which depend on N kinematic variables $x = (x_1, ..., x_N)$. We view the kinematic variables as coordinates on the kinematic space X. Let $I = (I_1, ..., I_n)^T$ be a vector of n master integrals. We assume that the master integrals satisfy an ε -factorised differential equation

$$dI(x,\mathbf{\varepsilon}) = \mathbf{\varepsilon} A(x)I(x,\mathbf{\varepsilon}). \tag{9}$$

with an integrable connection A:

$$dA = 0 \quad \text{and} \quad A \wedge A = 0. \tag{10}$$

In addition, we will always assume that when we restrict the kinematic variables to a sub-space, where the geometry reduces to a curve of genus zero, the master integrals will be pure¹. The

¹This excludes for example the basis K from ref. [16].

differential d is the differential in the kinematic variables x:

$$d = \sum_{j=1}^{N} dx_j \frac{\partial}{\partial x_j}.$$
 (11)

The condition dA = 0 states that the entries of the $(n \times n)$ -matrix A are closed one-forms. We denote a basis of differential one-forms appearing in A by $\omega_1, \ldots, \omega_{N_L}$ and the \mathbb{C} -vector space they span by $\Omega^1(X)$. We have

$$\dim \Omega^1(X) = N_L. \tag{12}$$

If we assume that the master integrals are ordered such that I_1 is the simplest and I_n the most complicated, then the matrix A has a lower block triangular structure induced by the sectors (or topologies) of the family of Feynman integrals under consideration. We distinguish blocks on the diagonal and blocks off the diagonal. A block on the diagonal corresponds to the maximal cut of the corresponding sector. If a family of Feynman integrals has s sectors then the matrix A is of the form

$$A = \begin{pmatrix} D_{1} & 0 & \dots & 0 \\ N_{21} & D_{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & D_{s-1} & 0 \\ N_{s1} & \dots & N_{s(s-1)} & D_{s} \end{pmatrix},$$
(13)

with diagonal blocks D_i and non-diagonal blocks N_{ij} .

At one loop, every sector has just one master integral, but starting from two loops we may have sectors with two or more master integrals. In this paper we are primarily interested in sectors of the second type.

2.2 Self-duality

Let us consider an $(n \times n)$ diagonal block

$$D = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1(n-1)} & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2(n-1)} & d_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ d_{(n-1)1} & d_{(n-1)2} & \dots & d_{(n-1)(n-1)} & d_{(n-1)n} \\ d_{n1} & d_{n2} & \dots & d_{n(n-1)} & d_{nn} \end{pmatrix}.$$
(14)

Self-duality is the statement

$$d_{ij} = d_{(n+1-j)(n+1-i)}. (15)$$

It corresponds to a reflection symmetry with respect to the anti-diagonal, as indicated by the colours in eq. (14).

Self-duality has been observed for the first time in the *l*-loop banana integrals of equal mass [10, 11, 17]. These integrals are related to Calabi–Yau geometries and the name derives from self-dual properties of Calabi–Yau operators [18]. However, the symmetry stated in eq. (15) is more general and not necessarily tied to Calabi–Yau geometries [12].

Self-duality is a symmetry of the diagonal blocks. There is no reason to expect self-duality to hold beyond the diagonal blocks if we keep the ordering: If we order the master integrals by sectors, and the sectors from the simplest to the most complicated, self-duality would relate the most complicated sector to the simplest one, which is obviously impossible.

2.3 Galois theory

Given a non-constant polynomial p(x) with coefficients from a field F, the roots of p(x) may not lie in F. In this case one considers the splitting field L/F, which is the smallest field extension that contains all the roots of p(x). The Galois group

$$G(L/F) = \{ \sigma \in \operatorname{Aut}(L) \mid \sigma|_F = \operatorname{id} \}$$
 (16)

is the subgroup of the automorphism group of L, which keeps F fixed.

A trivial example is given by the polynomial $p(x) = x^2 - 3 \in \mathbb{Q}[x]$. The roots of p(x) lie in $\mathbb{Q}[\sqrt{3}]$ and the Galois group is

$$G\left(\mathbb{Q}[\sqrt{3}]/\mathbb{Q}\right) = \mathbb{Z}_2,\tag{17}$$

generated by

$$\sigma : \mathbb{Q}[\sqrt{3}] \to \mathbb{Q}[\sqrt{3}],$$

$$\sigma\left(\sqrt{3}\right) = -\sqrt{3}.$$
(18)

In the application towards Feynman integrals we often encounter roots r of quadratic equations, where the Galois group acts as $r \to -r$. A typical example is the square root

$$r = \sqrt{-s(4m^2 - s)}. (19)$$

In the differential equation we will have differential one-forms that are even, like

$$\omega_0 = d \ln \left(\frac{s}{\mu^2} \right) \tag{20}$$

and differential one-forms that are odd, like

$$\omega_1 = \frac{1}{2} d \ln \left(\frac{2m^2 - s - r}{2m^2 - s + r} \right).$$
(21)

The group element σ sending r to (-r) acts on these as

$$\sigma(\omega_0) = \omega_0, \quad \sigma(\omega_1) = -\omega_1.$$
 (22)

If two master integrals J_1 and J_2 are related by $J_2 = \sigma(J_1)$ with $\sigma^2 = \mathrm{id}$ it follows from

$$dJ_1 = \varepsilon \sum_{j=1}^n a_{1j} J_j, \quad dJ_2 = \varepsilon \sum_{j=1}^n a_{2j} J_j$$
 (23)

that

$$\sigma(a_{12}) = a_{21}, \quad \sigma(a_{11}) = a_{22}.$$
 (24)

If in addition the remaining master integrals are invariant under the action of σ , i.e. $\sigma(J_k) = J_k$ for $k \in \{3, ..., n\}$ we further have

$$\sigma(a_{1k}) = a_{2k} \text{ for } k \in \{3, ..., n\}.$$
 (25)

More formally, we may view a system of Feynman integrals as a vector bundle. The base space is parameterised by the kinematic variables x, and for each point in the base space we have a vector space in the fibre. This vector space is spanned by the master integrals. Initially we may take this vector space to be defined over the field $\mathbb{Q}(x,\varepsilon)$, the field of rational functions with rational coefficients in the kinematic variables x and the dimensional regularisation parameter ε . In a pre-canonical basis this is all what is needed: In the differential equation for this basis we will only have rational functions in x and ε . However, we are interested in an ε -factorised basis and this may require to enlarge the field, for example by adjoining a root r. In this case we are led to a vector space over the field $\mathbb{Q}(x,\varepsilon)[r]$. Throughout this paper we will assume that any element of the Galois group acts trivially on any pre-canonical master integral:

$$\sigma(K) = K, \tag{26}$$

where σ is an element of the Galois group and K a pre-canonical master integral.

2.4 Combination of Galois symmetries and self-duality

Given a sector with two master integrals it is always possible to impose a Galois symmetry (without requiring in addition self-duality): For two master integrals $I = (I_1, I_2)^T$ satisfying the ε -factorised differential equation

$$dI = \varepsilon \tilde{A}I, \qquad \tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix}, \tag{27}$$

one sets

$$J_1 = I_1 + rI_2,$$

 $J_2 = I_1 - rI_2,$ (28)

where r is an algebraic extension of \mathbb{Q} , for example r = i. We define the Galois action by $\sigma(r) = -r$, and σ acts trivially on all other expressions. Clearly we have

$$J_2 = \sigma(J_1). \tag{29}$$

In the new basis $J = (J_1, J_2)^T$ the differential equation reads

$$dJ = \varepsilon AJ \tag{30}$$

with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \tilde{a}_{11} + \tilde{a}_{22} + r^{-1} \tilde{a}_{12} + r \tilde{a}_{21} & \tilde{a}_{11} - \tilde{a}_{22} - r^{-1} \tilde{a}_{12} + r \tilde{a}_{21} \\ \tilde{a}_{11} - \tilde{a}_{22} + r^{-1} \tilde{a}_{12} - r \tilde{a}_{21} & \tilde{a}_{11} + \tilde{a}_{22} - r^{-1} \tilde{a}_{12} - r \tilde{a}_{21} \end{pmatrix}.$$
(31)

The matrix A has the Galois symmetries

$$a_{22} = \sigma(a_{11}), \quad a_{21} = \sigma(a_{12}).$$
 (32)

However, A is in general not self-dual since

$$a_{11} \neq \sigma(a_{11}).$$
 (33)

Self-duality of A will require

$$\tilde{a}_{12} + r^2 \tilde{a}_{21} = 0. ag{34}$$

The possibility of finding a basis which makes self-duality and Galois symmetries manifest is therefore a non-trivial property. As Galois symmetry alone is trivial, we are always interested in the case where we might have Galois symmetries in addition to self-duality.

Let us summarise: For a sector with two master integrals it is often possible to find a basis $J = (J_1, J_2)^T$ such that

$$dJ = \varepsilon AJ, \tag{35}$$

and A has the structure

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{36}$$

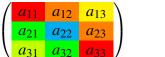
where entries with the same background colour are related by a symmetry. Self-duality relates

$$a_{11} = a_{22}.$$
 (37)

Furthermore, there is a \mathbb{Z}_2 -group with a generator σ such that

$$a_{11} = \sigma(a_{11}), \quad a_{12} = \sigma(a_{21}), \quad a_{21} = \sigma(a_{12}), \quad a_{22} = \sigma(a_{22}).$$
 (38)

This is the Galois symmetry. We see that only two of the four entries of the (2×2) -matrix need to be known, the other two follow from symmetries.





$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Figure 1: The effect of various symmetries on a (3×3) diagonal block: Entries with the same background colour are related by a symmetry. Left: Self-duality symmetry. Middle: The Galois symmetry $I_3 = \sigma(I_1)$, $I_2 = \sigma(I_2)$. Right: The combination of both.

It is worth discussing these concepts for sectors with more than two master integrals. Let us consider a sector with three master integrals $I = (I_1, I_2, I_3)^T$ and the ε -factorised differential equation

$$dI = \varepsilon AI, \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{39}$$

Self-duality gives the three relations

$$a_{33} = a_{11}, \quad a_{23} = a_{12}, \quad a_{32} = a_{21}.$$
 (40)

Let us assume that there exists a Galois symmetry, which relates I_1 and I_3

$$I_3 = \sigma(I_1), \tag{41}$$

and which acts trivially on I_2 :

$$I_2 = \sigma(I_2). \tag{42}$$

The Galois symmetry alone leads to the relations

$$a_{31} = \sigma(a_{13}), \quad a_{32} = \sigma(a_{12}), \quad a_{33} = \sigma(a_{11}), \quad a_{21} = \sigma(a_{23}),$$
 (43)

and to the invariance relation $a_{22} = \sigma(a_{22})$. Combining self-duality and Galois symmetry we obtain the relations

$$a_{33} = a_{11}, \quad a_{23} = a_{12}, \quad a_{32} = a_{21} = \sigma(a_{12}), \quad a_{31} = \sigma(a_{13}),$$
 (44)

and

$$a_{11} = \sigma(a_{11}), \quad a_{22} = \sigma(a_{22}).$$
 (45)

This is illustrated in fig. 1. Combining self-duality and Galois symmetry, only four out of the nine entries of the matrix *A* need to be known, the remaining ones follow from symmetry.

The Galois symmetries extend beyond the maximal cut. To discuss this point let us consider a system consisting of two sectors with two master integrals each, satisfying an ε -factorised differential equation

$$dI = \varepsilon AI, \quad A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}. \tag{46}$$

If the master integrals have been chosen such that self-duality is manifest, we have

$$a_{22} = a_{11}$$
 and $a_{44} = a_{33}$. (47)

Let us further assume that there are Galois group elements σ and σ' , which relate I_1, I_2 and I_3, I_4 , respectively. In other words,

$$I_2 = \sigma(I_1)$$
 and $I_4 = \sigma'(I_3)$. (48)

We then have

$$a_{21} = \sigma(a_{12}), \quad a_{43} = \sigma'(a_{34}).$$
 (49)

Two cases are relevant: Within the first case, σ acts trivially on I_3 , I_4

$$\sigma(I_3) = I_3 \text{ and } \sigma(I_4) = I_4,$$
 (50)

and σ' acts trivially on I_1, I_2

$$\sigma'(I_1) = I_1 \text{ and } \sigma'(I_2) = I_2.$$
 (51)

In this case the entries of the lower-left non-diagonal block are related as

$$a_{32} = \sigma(a_{31}), \qquad a_{41} = \sigma'(a_{31}), \qquad a_{42} = \sigma'(\sigma(a_{31})).$$
 (52)

We see that in this case we only need to know one entry of the non-diagonal block, the other three follow from symmetry. In summary, we have in this case the following structure of the matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$
 (53)

where entries with the same background colour are related by a symmetry.

The second relevant case is $\sigma = \sigma'$, i.e. the group element relates I_1 to I_2 as well as I_3 to I_4 . We look again at the lower-left non-diagonal block. In this case we have

$$a_{42} = \sigma(a_{31}), \quad a_{41} = \sigma(a_{32}),$$
 (54)

and two entries of the non-diagonal block may be obtained from Galois symmetries. The structure of the matrix A is then

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$
 (55)

where again entries with the same background colour are related by a symmetry.

The non-diagonal blocks need not be square matrices. In the case where we have one sector with one master integral and a second sector with two master integrals together with a generator σ of the Galois group, the structure of the matrix A in a suitable basis is given by

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{56}$$

Self-duality relates $a_{33} = a_{22}$, the Galois symmetry relates on the diagonal block $a_{32} = \sigma(a_{23})$. On the (2×1) -non-diagonal block the Galois symmetry relates

$$a_{31} = \sigma(a_{21}).$$
 (57)

The situation is similar if we have one sector with two master integrals and one sector with one master integral (ordered from the simplest to the most complicated sector). The structure of the matrix *A* in a suitable basis is

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{58}$$

Self-duality relates $a_{22} = a_{11}$, the Galois symmetry relates on the diagonal block $a_{21} = \sigma(a_{12})$. On the (1×2) -non-diagonal block the Galois symmetry relates

$$a_{32} = \sigma(a_{31}). (59)$$

2.5 Galois symmetries and rationalisations

Certain square roots can be rationalised [19,20] and one may consider the fate of Galois symmetries if one does so. We discuss this case with a simple example, involving the square root

$$r = \sqrt{-v(4-v)}. (60)$$

Typical differential one-forms in this case are

$$\omega_1 = d \ln(-v) \quad \omega_2 = d \ln(4-v), \quad \omega_3 = \frac{1}{2} d \ln \frac{2-v-r}{2-v+r}.$$
 (61)

Let σ be the element of the Galois group, which sends $r \to -r$. The first two differential one-forms are even under σ , the third one is odd:

$$\sigma(\omega_1) = \omega_1, \quad \sigma(\omega_2) = \omega_2, \quad \sigma(\omega_3) = -\omega_3.$$
 (62)

The root r is rationalised by

$$v = -\frac{(1-x)^2}{x}, \quad x = \frac{1}{2}(2-v-r), \quad r = \frac{1-x^2}{x}.$$
 (63)

In the variable x we have

$$\omega_1 = 2d \ln(1-x) - d \ln x, \quad \omega_2 = 2d \ln(1+x) - d \ln x, \quad \omega_3 = d \ln x.$$
 (64)

The transformation $\sigma(r) = -r$ translates to $\sigma(x) = x^{-1}$. It is easily checked that ω_1 and ω_2 are invariant under $x \to x^{-1}$, while ω_3 changes the sign under this transformation². Thus we see that the automorphism σ of $\mathbb{Q}(v, \varepsilon)[r]$, which keeps $\mathbb{Q}(v, \varepsilon)$ fixed and sends $r \to -r$ corresponds to the automorphism $x \to x^{-1}$ of $\mathbb{Q}(x, \varepsilon)$. Note that in the latter case there is no Galois extension.

2.6 Parameterised Galois symmetries and limit Galois symmetries

In some cases the transformation that realises self-duality and Galois symmetries is not unique. For example, it might occur that the transformation

$$J_1 = I_1 + rI_2, \quad r = \sqrt{\lambda},$$

 $J_2 = I_1 - rI_2,$ (65)

realises self-duality and Galois symmetry for any value $\lambda \in \mathbb{Q}$ that is not a perfect square. We call this a parameterised Galois symmetry. An example will be given in section 3.6. We have to exclude the case where λ is a perfect square. If λ is a perfect square, we have $r \in \mathbb{Q}$ and there is no field extension. Furthermore, there exists for $r \in \mathbb{Q}$ no field automorphism of \mathbb{Q} which sends r to -r.

Nevertheless, it will be useful to introduce something which comes close to being a Galois symmetry with respect to a square root of a perfect square, as there will be cases where there is no (normal) Galois symmetry on top of self-duality. We introduce the concept of a limit Galois symmetry as follows: We first divide the rational numbers into

$$\mathbb{Q} = \mathbb{PS} \cup \mathbb{NPS}, \tag{66}$$

where \mathbb{PS} denotes the set of rational numbers that are perfects squares and \mathbb{NPS} the set of rational numbers that are not perfect squares. We consider sequences

$$(\lambda_n) \in \mathbb{NPS}$$
 (67)

The condition $\sigma(r) = -r$ leads in x-space to the two possible transformations $x \to x^{-1}$ and $x \to -x$. However, only the first one leaves ω_1 and ω_2 invariant.

with

$$\lim_{n\to\infty}\lambda_n = \lambda \in \mathbb{PS}. \tag{68}$$

For each such sequence and each λ_n we consider the field extension $\mathbb{Q}(x, \varepsilon)[\sqrt{\lambda_n}]$ and redefine the master integrals for example as

$$J_1^{(n)} = I_1 + \sqrt{\lambda_n} I_2, \quad J_2^{(n)} = I_1 - \sqrt{\lambda_n} I_2.$$
 (69)

For any $\lambda_n \in \mathbb{NPS}$ the master integrals $J_1^{(n)}$ and $J_2^{(n)}$ are related by a Galois symmetry

$$J_2^{(n)} = \sigma\left(J_1^{(n)}\right),\tag{70}$$

where σ sends $\sqrt{\lambda_n}$ to $-\sqrt{\lambda_n}$. We further set

$$J_{1} = \lim_{n \to \infty} J_{1}^{(n)} = I_{1} + \sqrt{\lambda} I_{2},$$

$$J_{2} = \lim_{n \to \infty} J_{2}^{(n)} = I_{1} - \sqrt{\lambda} I_{2}.$$
(71)

We say that J_1 and J_2 are related by a limit Galois symmetry, i.e. for any sequence (λ_n) of non-perfect squares which converges to the perfect square λ we have for any λ_n a Galois symmetry in the usual sense.

Our most important example will be a sequence (λ_n) which converges to 1. We may take

$$\lambda_n = 1 - \frac{1}{p_n},\tag{72}$$

where p_n denotes the *n*-th prime number. Clearly, λ_n is not a perfect square and $\lim_{n\to\infty} \lambda_n = 1$. It will be convenient to use a short-hand notation for a limit Galois symmetry, mirroring the one we use for a normal Galois symmetry. In the following we will for example simply write

$$J_1 = I_1 + rI_2, \quad J_2 = I_1 - rI_2, \quad r = \sqrt{1},$$
 (73)

where $\sqrt{1}$ is understood in the sense discussed above. In the combination of self-duality and a limit Galois symmetry we will require self-duality only in the limit $n \to \infty$. In section 3.5 we will present an example where there is no (normal) Galois symmetry in addition to self-duality. However, there is a limit Galois symmetry in addition to self-duality.

3 Examples with dlog-forms

In this section we show in several examples that it is often possible to realise self-duality and Galois symmetries. For now we limit the discussion to Feynman integrals whose ε -factorised differential equation involves only dlog-forms with algebraic arguments. Typically we will have square roots appearing in the arguments of the dlog-forms. For the study of symmetries it is irrelevant whether the square roots can be rationalised simultaneously or not.

Examples of Feynman integrals with ε-factorised differential equations beyond dlog-forms will be discussed in section 4.

3.1 Drell-Yan

3.1.1 **Set-up**

As our main example, which we discuss at length, serves a diagram contributing to the mixed QCD-electroweak corrections to the Drell-Yan process. This example involves three square roots. It is known that the three square roots cannot be rationalised simultaneously [21]. However, the result can be expressed in terms of multiple polylogarithms [22]. The example we discuss is the minimal example with three non-simultaneously rationalisable square roots. This section also serves to set-up our notation for all further examples.

We consider the integrals

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9} = e^{2\gamma_E \epsilon} \left(\mu^2\right)^{\nu-D} \int \left(\prod_{a=1}^2 \frac{d^D k_a}{i\pi^{\frac{D}{2}}}\right) \left(\prod_{c=1}^9 \frac{1}{P_c^{\nu_c}}\right),\tag{74}$$

where D denotes the number of space-time dimensions, ε the dimensional regularisation parameter, γ_E the Euler–Mascheroni constant and

$$v = \sum_{j=1}^{9} v_j. \tag{75}$$

The inverse propagators are given by

$$P_{1} = -k_{2}^{2}, P_{2} = -k_{1}^{2} + m^{2}, P_{3} = -(k_{1} - k_{2})^{2},$$

$$P_{4} = -(k_{2} + p_{1})^{2}, P_{5} = -(k_{1} + p_{1})^{2}, P_{6} = -(k_{2} + p_{1} + p_{2})^{2},$$

$$P_{7} = -(k_{1} + p_{1} + p_{2})^{2} + m^{2}, P_{8} = -(k_{2} - p_{4})^{2}, P_{9} = -(k_{1} - p_{4})^{2}. (76)$$

The external particles are assumed to be massless: $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$. The Mandelstam variables s and t are defined by

$$s = (p_1 + p_2)^2, t = (p_2 + p_3)^2.$$
 (77)

A sector is defined by the set of propagators with positive exponents. We define the sector id by

ID =
$$\sum_{j=1}^{9} 2^{j-1} \Theta(\mathbf{v}_j)$$
, (78)

with $\Theta(x) = 1$ for x > 0 and $\Theta(x) = 0$ for $x \le 0$. We further define the dimension-shift operator \mathbf{D}^- , which lowers the dimension of space-time by two units through

$$\mathbf{D}^{-}I_{\nu_{1}\nu_{2}\nu_{3}\nu_{4}\nu_{5}\nu_{6}\nu_{7}\nu_{8}\nu_{9}}(D) = I_{\nu_{1}\nu_{2}\nu_{3}\nu_{4}\nu_{5}\nu_{6}\nu_{7}\nu_{8}\nu_{9}}(D-2). \tag{79}$$

We consider the sector 215 with the propagators $P_1, P_2, P_3, P_5, P_7, P_8$. The Feynman diagram for

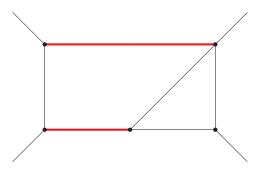


Figure 2: The Feynman diagram for the sector 215. Black lines denote massless particles, red lines denote particles with a mass m.

this sector is shown in fig. 2. The system has 16 master integrals. It is the minimal example with non-simultaneously rationalisable square roots. A pre-canonical basis is given by

$$I_{111000000},\,I_{101000100},\,I_{1(-1)1000100},\,I_{001010010},\,I_{111000100},\,I_{011010010},\,I_{011000110},\,I_{(-1)11000110},\\I_{111010100},\,I_{111010010},\,I_{111000110},\,I_{101010110},\,I_{1(-1)1010110},\,I_{011010110},\,I_{(-1)11010110},\,I_{111010110}.\\$$

An overview of the master integrals is given in table 1. There are four sectors (sectors 69, 198, 213 and 214) with two master integrals. We may transform to a basis $I = (I_1, ..., I_{16})^T$, which puts the differential equation into an ε -factorised form at the expense of introducing three square roots (see ref. [22]):

$$r_{1} = \sqrt{-s(4m^{2}-s)},$$

$$r_{2} = \sqrt{(-s)(-t)[4m^{2}(-t-m^{2})-s(-t)]},$$

$$r_{3} = \sqrt{(-s)[(4m^{2}-s)(-t)^{2}+2m^{2}(-s)(-t)+m^{4}(-s)]}.$$
(81)

Such a basis is given by

Sector 7:
$$I_1 = \varepsilon^2 \frac{m^2}{\mu^2} \mathbf{D}^- I_{111000000},$$

Sector 69: $I_2 = \varepsilon^2 \frac{(m^2 - s)}{\mu^2} \mathbf{D}^- I_{101000100},$
 $I_3 = \varepsilon^2 \left(\mathbf{D}^- I_{1(-1)1000100} - \frac{m^2}{\mu^2} \mathbf{D}^- I_{101000100} \right),$
Sector 148: $I_4 = \varepsilon^2 \frac{t}{\mu^2} \mathbf{D}^- I_{001010010},$
Sector 71: $I_5 = \varepsilon^2 \frac{r_1}{\mu^2} \left(\frac{m^2}{\mu^2} \mathbf{D}^- I_{111000100} - \mathbf{D}^- I_{101000100} \right),$

number of	block	sector	master integrals	master integrals	roots
propagators			basis \vec{I}	basis \vec{J}	
3	1	7	I_1	J_1	
	2	69	I_2, I_3	J_2, J_3	
	3	148	I_4	J_4	
4	4	71	I_5	J_5	r_1
	5	150	I_6	J_6	
	6	198	I_7, I_8	J_7, J_8	r_1
5	7	87	<i>I</i> ₉	J_9	
	8	151	I_{10}	J_{10}	
	9	199	I_{11}	J_{11}	
	10	213	I_{12}, I_{13}	J_{12}, J_{13}	
	11	214	I_{14}, I_{15}	$J_{12}, J_{13} J_{14}, J_{15}$	r_2
6	12	215	I_{16}	J_{16}	<i>r</i> ₃

Table 1: Overview of the set of master integrals. The first column denotes the number of propagators, the second column labels consecutively the sectors, the third column gives the sector id (defined in eq. (78)), the fourth column lists the master integrals in the basis \vec{I} , the fifth column the corresponding ones in the basis \vec{J} . The last column denotes the dependence on square roots.

Sector 150:
$$I_{6} = \varepsilon^{3} \frac{t}{\mu^{2}} I_{012010010},$$

Sector 198: $I_{7} = \varepsilon^{3} \frac{s}{\mu^{2}} I_{012000110},$
 $I_{8} = r_{1} \left(\frac{1}{\varepsilon} \frac{\partial}{\partial s} - \frac{1}{s} \right) I_{7},$
Sector 87: $I_{9} = \varepsilon^{3} \frac{m^{2}s}{\mu^{4}} I_{211010100},$
Sector 151: $I_{10} = \varepsilon^{4} \frac{t}{\mu^{2}} I_{111010010},$
Sector 199: $I_{11} = \varepsilon^{4} \frac{s}{\mu^{2}} I_{111000110},$
Sector 213: $I_{12} = \varepsilon^{4} \frac{(s+t)}{\mu^{2}} I_{101010110},$
 $I_{13} = \varepsilon^{3} \frac{m^{2}(s+t)}{\mu^{4}} I_{101010210},$
Sector 214: $I_{14} = \varepsilon^{3} (1 - 2\varepsilon) \frac{s}{\mu^{2}} I_{011010110},$
 $I_{15} = \varepsilon^{3} \frac{r_{2}}{\mu^{4}} I_{012010110},$

Sector 215:
$$I_{16} = \varepsilon^4 \frac{r_3}{\mu^4} I_{111010110}$$
. (82)

In this basis we have an ε -factorised differential equation

$$dI = \varepsilon \tilde{A}I. \tag{83}$$

We write the differential one-forms appearing in A as

$$\omega_j = d \ln l_j. \tag{84}$$

We call the l_j 's letters. In total there are 17 letters³, which can be divided into rational letters and non-rational letters. The rational letters are

$$l_{1} = \frac{m^{2}}{\mu^{2}}, \qquad l_{2} = \frac{-s}{\mu^{2}}, \qquad l_{3} = \frac{-t}{\mu^{2}},$$

$$l_{4} = \frac{m^{2} - s}{\mu^{2}}, \qquad l_{5} = \frac{4m^{2} - s}{\mu^{2}}, \qquad l_{6} = \frac{m^{2} + t}{\mu^{2}},$$

$$l_{7} = \frac{-s - t}{\mu^{2}}, \qquad l_{8} = \frac{4m^{2}(-t - m^{2}) + st}{\mu^{4}}, \qquad l_{9} = \frac{4m^{2}t^{2} - s(m^{2} - t)^{2}}{\mu^{6}}.$$
(85)

The non-rational letters are

$$l_{10} = \frac{2m^{2} - s - r_{1}}{2m^{2} - s + r_{1}}, \qquad l_{11} = \frac{(2m^{2} - s)(-t) - r_{2}}{(2m^{2} - s)(-t) + r_{2}},$$

$$l_{12} = \frac{(-s)(-t) - r_{2}}{(-s)(-t) + r_{2}}, \qquad l_{13} = \frac{(-s)\left[(4m^{2} - s)(-t) - 2m^{4}\right] - r_{1}r_{2}}{(-s)\left[(4m^{2} - s)(-t) - 2m^{4}\right] + r_{1}r_{2}},$$

$$l_{14} = \frac{(-s)(m^{2} - t) - r_{3}}{(-s)(m^{2} - t) + r_{3}}, \qquad l_{15} = \frac{(-s)(m^{2} + t) - r_{3}}{(-s)(m^{2} + t) + r_{3}},$$

$$l_{16} = \frac{(-s)(st - m^{2}s - 4m^{2}t) - r_{1}r_{3}}{(-s)(st - m^{2}s - 4m^{2}t) + r_{1}r_{3}}, \qquad l_{17} = \frac{st(st - m^{2}s - 4m^{2}t - 2m^{4}) - r_{2}r_{3}}{st(st - m^{2}s - 4m^{2}t - 2m^{4}) + r_{2}r_{3}}. \qquad (86)$$

In the basis I the diagonal blocks \tilde{D}_i and the non-diagonal blocks \tilde{N}_{ij} do not have any particular structure. We give three examples: The diagonal block for sector 198 is given by

$$\tilde{D}_{6} = \begin{pmatrix} -3\omega_{1} + \omega_{2} & \frac{1}{2}\omega_{10} \\ \frac{3}{2}\omega_{10} & -\omega_{1} - \omega_{5} \end{pmatrix}, \tag{87}$$

the diagonal block for sector 214 is given by

$$\tilde{D}_{11} = \begin{pmatrix} -3\omega_1 + 2\omega_2 - \frac{1}{2}\omega_3 - \frac{1}{2}\omega_7 & \frac{1}{2}\omega_{11} \\ -\frac{3}{2}\omega_{11} & \omega_1 - \frac{1}{2}\omega_3 + \frac{3}{2}\omega_7 - 2\omega_8 \end{pmatrix}, \tag{88}$$

and the non-diagonal block $\tilde{N}_{(11)6}$ is given by

$$\tilde{N}_{(11)6} = \begin{pmatrix} -\frac{1}{2}\omega_3 + \frac{1}{2}\omega_7 & -\frac{1}{2}\omega_{10} \\ \frac{3}{2}\omega_{11} & -\frac{1}{2}\omega_{13} \end{pmatrix}. \tag{89}$$

³Setting for example $\mu = m$ reduces the number of letters by one. In this case we have $d \ln l_1 = 0$.

3.1.2 A refined basis and the Galois group

For the Drell-Yan example we may achieve the symmetries of eq. (37), eq. (38), eq. (52) and eq. (54) for all sectors by a change to a new basis $J = (J_1, \dots, J_{16})^T$ defined by

Sector 69:
$$J_2 = I_2 + \sqrt{3}I_3$$
, $J_3 = I_2 - \sqrt{3}I_3$,
Sector 198: $J_7 = I_7 + \frac{i}{3}\sqrt{3}I_8$, $J_8 = I_7 - \frac{i}{3}\sqrt{3}I_8$,
Sector 87: $J_9 = I_9 - \frac{1}{8}(I_2 + I_3)$,
Sector 199: $J_{11} = I_{11} + \frac{1}{2}I_7$,
Sector 213: $J_{12} = \sqrt{3}I_{12} - \left(1 + \sqrt{3}\right)I_{13}$, $J_{13} = -\sqrt{3}I_{12} - \left(1 - \sqrt{3}\right)I_{13}$,
Sector 214: $J_{14} = 2I_{14} + \frac{2}{3}\sqrt{3}I_{15}$, $J_{15} = 2I_{14} - \frac{2}{3}\sqrt{3}I_{15}$, (90)

and $J_k = I_k$ for all other master integrals. Apart from the three square roots r_1, r_2, r_3 we have two new (trivial) square roots

$$s_3 = \sqrt{3}, \quad s_{-3} = i\sqrt{3}.$$
 (91)

The square roots s_3 and s_{-3} do not depend on the kinematic variables. The Galois group is

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{92}$$

with generators $\rho_1, \rho_2, \rho_3, \sigma_3, \sigma_{-3}$. We adopt the convention that we label square roots, which depend on kinematic variables by r_i , and square roots, which do not depend on kinematic variables by s_i . The corresponding generators of the Galois group are denoted by ρ_i and σ_i , respectively. A generator ρ_i acts on r_i as

$$\rho_i(r_i) = -r_i, \tag{93}$$

and trivially on all other square roots

$$\rho_i(r_j) = r_j \quad \text{for } i \neq j \quad \text{and} \quad \rho_i(s_k) = s_k.$$
(94)

The action of σ_i is analogous:

$$\sigma_i(s_i) = -s_i \tag{95}$$

and trivially on all other square roots. In this new basis we have again an ε -factorised differential equation

$$dJ = \varepsilon AJ. \tag{96}$$

The individual blocks are now symmetric, for example the diagonal block for sector 198 is now given by

$$D_6 = \begin{pmatrix} -2\omega_1 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_5 & -\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_5 + \frac{i\sqrt{3}}{2}\omega_{10} \\ -\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_5 - \frac{i\sqrt{3}}{2}\omega_{10} & -2\omega_1 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_5 \end{pmatrix},$$
(97)

We have

$$d_{11}^{(6)} = \rho_{1}\left(d_{11}^{(6)}\right) = \sigma_{-3}\left(d_{11}^{(6)}\right) = \rho_{1}\left(d_{22}^{(6)}\right) = \sigma_{-3}\left(d_{22}^{(6)}\right) = d_{22}^{(6)},$$

$$d_{21}^{(6)} = \rho_{1}\left(d_{12}^{(6)}\right) = \sigma_{-3}\left(d_{12}^{(6)}\right),$$
(98)

since

$$\rho_1(\omega_{10}) = -\omega_{10}. \tag{99}$$

For the diagonal block of sector 214 we find

$$D_{11} = \begin{pmatrix} -\omega_1 + \omega_2 - \frac{1}{2}\omega_3 + \frac{1}{2}\omega_7 - \omega_8 & -2\omega_1 + \omega_2 - \omega_7 + \omega_8 - \frac{\sqrt{3}}{2}\omega_{11} \\ -2\omega_1 + \omega_2 - \omega_7 + \omega_8 + \frac{\sqrt{3}}{2}\omega_{11} & -\omega_1 + \omega_2 - \frac{1}{2}\omega_3 + \frac{1}{2}\omega_7 - \omega_8 \end{pmatrix}. (100)$$

We have (with $\rho_2(\omega_{11}) = -\omega_{11}$)

$$d_{11}^{(11)} = \rho_2 \left(d_{11}^{(11)} \right) = \sigma_3 \left(d_{11}^{(11)} \right) = \rho_2 \left(d_{22}^{(11)} \right) = \sigma_3 \left(d_{22}^{(11)} \right) = d_{22}^{(11)},$$

$$d_{21}^{(11)} = \rho_2 \left(d_{12}^{(11)} \right) = \sigma_3 \left(d_{12}^{(11)} \right). \tag{101}$$

For the non-diagonal block $N_{(11)6}$ we find

$$N_{(11)6} = \begin{pmatrix} -\frac{1}{2}\omega_{3} + \frac{1}{2}\omega_{7} + \frac{i\sqrt{3}}{2}\omega_{10} + \frac{\sqrt{3}}{2}\omega_{11} + \frac{i}{2}\omega_{13} & -\frac{1}{2}\omega_{3} + \frac{1}{2}\omega_{7} - \frac{i\sqrt{3}}{2}\omega_{10} + \frac{\sqrt{3}}{2}\omega_{11} - \frac{i}{2}\omega_{13} \\ -\frac{1}{2}\omega_{3} + \frac{1}{2}\omega_{7} + \frac{i\sqrt{3}}{2}\omega_{10} - \frac{\sqrt{3}}{2}\omega_{11} - \frac{i}{2}\omega_{13} & -\frac{1}{2}\omega_{3} + \frac{1}{2}\omega_{7} - \frac{i\sqrt{3}}{2}\omega_{10} - \frac{\sqrt{3}}{2}\omega_{11} + \frac{i}{2}\omega_{13} \end{pmatrix}.$$

$$(102)$$

Here we have

$$\begin{split} n_{12}^{((11)6)} &= \rho_1 \left(n_{11}^{((11)6)} \right) = \sigma_{-3} \left(n_{11}^{((11)6)} \right), \\ n_{21}^{((11)6)} &= \rho_2 \left(n_{11}^{((11)6)} \right) = \sigma_3 \left(n_{11}^{((11)6)} \right), \\ n_{22}^{((11)6)} &= \rho_1 \rho_2 \left(n_{11}^{((11)6)} \right) = \rho_1 \sigma_3 \left(n_{11}^{((11)6)} \right) = \sigma_{-3} \rho_2 \left(n_{11}^{((11)6)} \right) = \sigma_{-3} \sigma_3 \left(n_{11}^{((11)6)} \right). \end{split}$$

Note that

$$i = \frac{1}{3}s_3s_{-3} \tag{104}$$

Sector	69	198	213	214
Galois symmetries	σ_3	ρ_1, σ_{-3}	σ_3	ρ_2, σ_3

Table 2: Overview of the Galois symmetries of the sectors with more than one master integral.

and therefore $\sigma_{-3}(i) = -i$. The Galois symmetries associated with the four sectors with two master integrals each are summarised in table 2. We have

$$J_{3} = \sigma_{3}(J_{2}),$$

$$J_{8} = \rho_{1}(J_{7}) = \sigma_{-3}(J_{7}),$$

$$J_{13} = \sigma_{3}(J_{12}),$$

$$J_{15} = \rho_{2}(J_{14}) = \sigma_{3}(J_{14}).$$
(105)

Note that for the sectors 198 and 214 there is more than one element from the Galois group that relates the two master integrals in this sector. Furthermore, note that the occurrence of a square root in a particular sector does not imply that this sector must have master integrals related by a Galois symmetry. Counterexamples are given by sector 71, where the definition of the master integral J_5 involves the square root r_1 and by sector 215, where the definition of the master integral J_{16} involves the square root r_3 . Both sectors have only one master integral.

In summary, the matrix A has the structure (for a compact notation we use hexadecimal indices)

Entries with the same background colour are related by a symmetry.

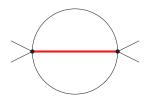


Figure 3: The Feynman diagram for the sector 69. Black lines denote massless particles, red lines denote particles with a mass m.

3.1.3 The simplest example

One of the simplest examples is given by the sector 69 of the previous family of Feynman integrals. It is worth discussing this example explicitly, as there is no square root with a dependence on the kinematic variables associated to it. This is the simplest example where the non-obvious square root $\sqrt{3}$ appears. The corresponding Feynman diagram is shown in fig. 3. This sector has no sub-sectors and forms a system with two master integrals, which we may takes as (I_2, I_3) , with I_2 and I_3 defined in eq. (82). The differential equation for these master integrals reads

$$d\begin{pmatrix} I_2 \\ I_3 \end{pmatrix} = \varepsilon \tilde{D}_2 \begin{pmatrix} I_2 \\ I_3 \end{pmatrix}, \tag{107}$$

with

$$\tilde{D}_2 = \begin{pmatrix} \frac{1}{2}\omega_1 + \frac{3}{2}\omega_2 - 4\omega_4 & -\frac{3}{2}\omega_1 + \frac{3}{2}\omega_2 \\ \frac{1}{2}\omega_1 - \frac{1}{2}\omega_2 & -\frac{3}{2}\omega_1 - \frac{1}{2}\omega_2 \end{pmatrix}.$$
(108)

The transformation from (I_2, I_3) to (J_2, J_3) given in eq. (90) converts the differential equation to

$$d\begin{pmatrix} J_2 \\ J_3 \end{pmatrix} = \varepsilon D_2 \begin{pmatrix} J_2 \\ J_3 \end{pmatrix} \tag{109}$$

with

$$D_{2} = \begin{pmatrix} -\frac{1}{2}\omega_{1} + \frac{1}{2}\omega_{2} - 2\omega_{4} & \left(1 + \frac{\sqrt{3}}{2}\right)\omega_{1} + \left(1 - \frac{\sqrt{3}}{2}\right)\omega_{2} - 2\omega_{4} \\ \left(1 - \frac{\sqrt{3}}{2}\right)\omega_{1} + \left(1 + \frac{\sqrt{3}}{2}\right)\omega_{2} - 2\omega_{4} & -\frac{1}{2}\omega_{1} + \frac{1}{2}\omega_{2} - 2\omega_{4} \end{pmatrix}. \quad (110)$$

The self-duality requirement

$$d_{11}^{(2)} = d_{22}^{(2)} (111)$$

will introduce the (non-obvious) square root $s_3 = \sqrt{3}$.

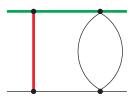


Figure 4: An example of a Feynman diagram relevant to electroweak corrections to electronnucleon scattering. Black lines denote massless particles, red lines denote particles with a mass m_Z , green lines denote particles with a mass m_N .

3.2 Electron-nucleon scattering

A second interesting example is the family of Feynman diagrams shown in fig. 4, contributing to the electroweak corrections to electron-nucleon scattering. This example has one sector with two master integrals and two square roots. The inverse propagators are now given by

$$P_{1} = -k_{1}^{2}, P_{2} = -(k_{1} - p_{1})^{2} + m_{Z}^{2}, P_{3} = -(k_{1} - p_{12})^{2} + m_{N}^{2},$$

$$P_{4} = -(k_{1} - p_{123})^{2}, P_{5} = -k_{2}^{2}, P_{6} = -(k_{1} + k_{2} - p_{123})^{2},$$

$$P_{7} = -(k_{1} + k_{2} + p_{12})^{2}, P_{8} = -(k_{1} + k_{2} - p_{1})^{2}, P_{9} = -(k_{1} + k_{2})^{2}, (112)$$

and the external momenta satisfy

$$p_1^2 = p_4^2 = 0, p_2^2 = p_3^2 = m_N^2.$$
 (113)

The Mandelstam variables s and t are defined as in eq. (77). In eq. (112) we used the short-hand notation $p_{12} = p_1 + p_2$ and $p_{123} = p_{12} + p_3$. A basis of master integrals, which puts the differential equation into an ε -factorised form was given in ref. [23] and reads

Sector 50:
$$I_{1}= \varepsilon^{2} \left(\frac{m_{Z}^{2}-t}{\mu^{2}}\right) \mathbf{D}^{-}I_{010011000},$$
 $I_{2}= \varepsilon^{2} \mathbf{D}^{-}I_{010(-1)11000},$
Sector 52: $I_{3}= \varepsilon (1+4\varepsilon) \left(\frac{m_{N}^{2}}{\mu^{2}}\right) \mathbf{D}^{-}I_{001011000},$
Sector 53: $I_{4}= -4\varepsilon^{3} \left(\frac{m_{N}^{2}-s}{\mu^{2}}\right) I_{101012000},$
Sector 54: $I_{5}= \varepsilon^{3} \left(\frac{r_{1}}{\mu^{2}}\right) I_{011012000},$
 $I_{6}= \varepsilon^{2} \left(\frac{r_{3}}{\mu^{2}}\right) \mathbf{D}^{-}I_{011(-1)11000},$
Sector 55: $I_{7}= \varepsilon^{3} \left(\frac{m_{N}^{2}-s}{\mu^{2}}\right) \left(\frac{m_{Z}^{2}-t}{\mu^{2}}\right) I_{111012000},$

Sector	50	54	55
Galois symmetries	σ_3	ρ_3,σ_{-3}	σ_3

Table 3: Overview of the Galois symmetries in the basis J of the sectors with more than one master integral for the example shown in fig. 4.

$$I_8 = \varepsilon^3 \left(\frac{m_N^2 - s}{\mu^2}\right) I_{111(-1)12000}. \tag{114}$$

There are three sectors with two master integrals (sectors 50, 54 and 55). Sector 50 in this example is an integral we encountered previously, it is the integral shown in fig. 3. Sector 54 introduces two square roots r_1 and r_3 , which are given by

$$r_1 = \sqrt{-t(4m_N^2 - t)}, \quad r_3 = \sqrt{-m_Z^2(4m_N^2 - m_Z^2)}.$$
 (115)

The occurrence of two square roots within one sector makes this example interesting. Changing the basis of master integrals to

Sector 50:
$$J_1 = I_1 + \sqrt{3}I_2$$
, $J_2 = I_1 - \sqrt{3}I_2$,
Sector 52: $J_3 = I_3$,
Sector 53: $J_4 = I_4$,
Sector 54: $J_5 = I_5 + \frac{i}{6}\sqrt{3}I_6$, $J_6 = I_5 - \frac{i}{6}\sqrt{3}I_6$,
Sector 55: $J_7 = I_7 + \sqrt{3}I_8$, $J_8 = I_7 - \sqrt{3}I_8$, (116)

and $J_k = I_k$ for all other master integrals will realise self-duality and the Galois symmetries. In eq. (116) we introduced again two square roots

$$s_3 = \sqrt{3}, \quad s_{-3} = i\sqrt{3}$$
 (117)

with no dependence on the kinematic variables. The Galois symmetries are summarised in table 3. We have

$$J_2 = \sigma_3(J_1),$$

 $J_6 = \rho_3(J_5) = \sigma_{-3}(J_5),$
 $J_8 = \sigma_3(J_7).$ (118)

In the basis J we have an ε -factorised differential equation

$$dJ = \varepsilon AJ \tag{119}$$

Sector	50	54	55
Galois symmetries	σ_3	ρ_1,σ_{-3}	σ_3

Table 4: Overview of the Galois symmetries in the basis J' of the sectors with more than one master integral for the example shown in fig. 4.

where A has the structure

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & a_{55} & a_{56} & 0 & 0 \\ a_{61} & a_{62} & a_{63} & 0 & a_{65} & a_{66} & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{pmatrix}.$$
 (120)

Entries with the same background colour are related by a symmetry.

The basis of master integrals that maximises the symmetries of the matrix A is not necessarily unique. To see this, let us discuss the roles of the roots r_1 and r_3 in this example. They appear in the same sector (sector 54 with master integrals J_5 and J_6). We have chosen a basis such that

$$\rho_3(J_5) = J_6. (121)$$

Acting with ρ_1 on J_5 gives us

$$\rho_1(J_5) = -J_6. (122)$$

Setting

$$J_5' = I_6 - 2i\sqrt{3}I_5,$$

$$J_6' = I_6 + 2i\sqrt{3}I_5$$
(123)

and $J'_k = J_k$ for all other integrals will reverse the roles of r_1 and r_3 . We now have

$$\rho_1(J_5') = J_6', \quad \rho_3(J_5') = -J_6'.$$
(124)

In the basis J' we have again an ε -factorised differential equation

$$dJ' = \varepsilon A'J'. \tag{125}$$

A' has the same structure as the one shown in eq. (120). The Galois symmetries in the basis J' are summarised in table 4.

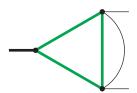


Figure 5: An example of a Feynman diagram contributing to the Higgs decay $H \to b\bar{b}$. Thin black lines denote massless particles, a thick black line denotes a particle with a mass m_H , green lines denote particles with a mass m_t .

3.3 An example with three master integrals within one sector

As an example with a sector with three master integrals we discuss a two-loop integral contributing to the Higgs decay $H \to b\bar{b}$, where the b-quarks are assumed to be massless. The diagram is shown in fig. 5. This family has in total five master integrals, grouped into three sectors. The first two sectors have just one master integral each. The first sector (with one master integral) is given by a product of two one-loop tadpole integrals, the second sector (again with one master integral) is given by the product of a one-loop two-point function with a one-loop tadpole integral. The third sector has three master integrals. We follow the notation of ref. [24]. The family of Feynman integrals has seven propagators

$$P_{1} = -k_{1}^{2} + m_{t}^{2}, P_{2} = -(k_{1} - p_{1} - p_{2})^{2} + m_{t}^{2}, P_{3} = -(k_{1} + k_{2})^{2},$$

$$P_{4} = -(k_{1} + k_{2} - p_{1})^{2}, P_{5} = -k_{2}^{2} + m_{W}^{2}, P_{6} = -(k_{2} + p_{2})^{2} + m_{t}^{2},$$

$$P_{7} = -(k_{1} - p_{1})^{2} + m_{t}^{2}. (126)$$

We set $s = (p_1 + p_2)^2$. There is one square root

$$r_1 = \sqrt{-s\left(4m_t^2 - s\right)}. (127)$$

A basis of master integrals, which puts the differential equation in ε-factorised form is given by⁴

$$I_{1} = \varepsilon^{2} \mathbf{D}^{-} I_{1000010},$$

$$I_{2} = \frac{1}{2} \varepsilon^{2} \frac{r_{1}}{\mu^{2}} \mathbf{D}^{-} I_{1100010},$$

$$I_{3} = \varepsilon^{3} \frac{s}{\mu^{2}} I_{1120010},$$

$$I_{4} = \varepsilon^{3} \frac{s}{\mu^{2}} I_{1110020},$$

$$I_{5} = \varepsilon^{2} \frac{r_{1}}{\mu^{2}} [(1 - 2\varepsilon) I_{2110010} + \varepsilon I_{1110020}].$$
(128)

⁴In ref. [24] these master integrals are denoted J_1 , J_3 , J_{17} , J_{18} and J_{19} .

The change of basis $J_1 = I_1$, $J_2 = I_2$ and

$$J_{3} = I_{3} + I_{4} + \frac{i\sqrt{3}}{3}I_{5},$$

$$J_{4} = \frac{i\sqrt{6}}{3}I_{3} + \frac{2i\sqrt{6}}{3}I_{4},$$

$$J_{5} = I_{3} + I_{4} - \frac{i\sqrt{3}}{3}I_{5},$$
(129)

introduces two constant square roots

$$s_{-3} = i\sqrt{3}, \quad s_{-6} = i\sqrt{6},$$
 (130)

and realises self-duality and the Galois symmetry. The differential equation reads

$$dJ = \varepsilon AJ, \ A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & 0 & a_{53} & a_{54} & a_{55} \end{pmatrix}. \tag{131}$$

Self-duality corresponds to the relations

$$a_{55} = a_{33}, \quad a_{45} = a_{34}, \quad a_{54} = a_{43}.$$
 (132)

The Galois symmetry $J_3 = \rho_1(J_2) = \sigma_{-3}(J_2)$ gives in addition the relations

$$a_{43} = \rho_1(a_{34}) = \sigma_{-3}(a_{34}),$$

 $a_{53} = \rho_1(a_{35}) = \sigma_{-3}(a_{35}),$
 $a_{51} = \rho_1(a_{31}) = \sigma_{-3}(a_{31}).$ (133)

It is worth noting that

$$\sqrt{2} = -\frac{1}{3}s_{-3}s_{-6}, \tag{134}$$

and therefore

$$\sigma_{-3}\left(\sqrt{2}\right) = -\sqrt{2} \text{ and } \sigma_{-3}\left(i\sqrt{6}\right) = i\sqrt{6}.$$
 (135)

The latter relation states that σ_{-3} acts on s_{-3} , but not on s_{-6} . If we set $\mu = m_t$, the entries of A' are linear combinations of

$$\omega_1 = \frac{ds}{s}, \quad \omega_2 = \frac{ds}{s - 4m_t^2}, \quad \omega_3 = \frac{ds}{\sqrt{-s\left(4m_t^2 - s\right)}} = \frac{1}{2}d\ln\frac{2m_t^2 - s - r_1}{2m_t^2 - s + r_1}.$$
(136)

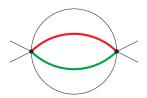


Figure 6: An example of a Feynman diagram contributing to the three-loop Higgs boson selfenergy. Black lines denote massless particles, red lines denote particles with a mass m_W , green lines denote particles with a mass m_t .

The differential one-form ω_3 is odd under ρ_1 :

$$\rho_1(\omega_3) = -\omega_3. \tag{137}$$

The entries a_{34} and a_{43} are given by

$$a_{34} = -\frac{i\sqrt{6}}{2}\omega_1 + \sqrt{2}\omega_3, \quad a_{43} = -\frac{i\sqrt{6}}{2}\omega_1 - \sqrt{2}\omega_3,$$
 (138)

and with eq. (135) and eq.(137) one directly verifies the relation $a_{43} = \rho_1(a_{34}) = \sigma_{-3}(a_{34})$. The symmetries of the diagonal block corresponding to sector 39 (the (3 × 3)-block formed by the master integrals J_3 , J_4 and J_5) follow the pattern of symmetries shown in fig. 1.

3.4 An example with four master integrals within one sector

As an advanced example we discuss a three-loop integral contributing to the Higgs boson self-energy. The diagram is shown in fig. 6. This family has a single sector with four master integrals. We follow the notation of ref. [25]. We consider the integrals

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9} = e^{3\gamma_E \varepsilon} \left(\mu^2\right)^{\nu - \frac{3}{2}D} \int \left(\prod_{a=1}^3 \frac{d^D k_a}{i\pi^{\frac{D}{2}}}\right) \left(\prod_{c=1}^9 \frac{1}{P_c^{\nu_c}}\right),\tag{139}$$

where the inverse propagators are given by

$$P_{1} = -k_{1}^{2} + m_{t}^{2}, P_{2} = -(k_{1} - p)^{2} + m_{t}^{2}, P_{3} = -(k_{1} + k_{2})^{2},$$

$$P_{4} = -k_{2}^{2} + m_{t}^{2}, P_{5} = -(k_{2} + k_{3})^{2} + m_{W}^{2}, P_{6} = -(k_{2} + p)^{2} + m_{t}^{2},$$

$$P_{7} = -k_{3}^{2}, P_{8} = -(k_{3} - p)^{2}, P_{9} = -(k_{1} - k_{3})^{2} + m_{t}^{2}. (140)$$

We are interested in sector 86 (with propagators 2,3,5,7). A basis of master integrals that puts the differential equation into an ε -factorised form is given by

$$I_1 = \varepsilon^3 \frac{r_2}{\mu^2} \mathbf{D}^- I_{011010100},$$

$$I_{2} = \varepsilon^{3} \left[\mathbf{D}^{-} I_{011(-1)10100} - \frac{\left(m_{t}^{2} - m_{W}^{2}\right)}{\mu^{2}} \mathbf{D}^{-} I_{011010100} \right],$$

$$I_{3} = \varepsilon^{3} \mathbf{D}^{-} I_{01101(-1)100},$$

$$I_{4} = \varepsilon^{3} \left[\mathbf{D}^{-} I_{0110101(-1)0} + \frac{p^{2}}{\mu^{2}} \mathbf{D}^{-} I_{011010100} \right].$$
(141)

In this example we have the square root of the Källen function

$$r_2 = \sqrt{\lambda \left(p^2, m_W^2, m_t^2\right)}. \tag{142}$$

We recall that the Källen function is defined by

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx. \tag{143}$$

The change of basis

$$J_{1} = \frac{i\sqrt{2}}{2}I_{1} + I_{2} + I_{3} + I_{4},$$

$$J_{2} = \frac{1}{2}\left(i + \sqrt{3}\right)I_{2} + \frac{1}{2}\left(i - \sqrt{3}\right)I_{3} - iI_{4},$$

$$J_{3} = \frac{1}{2}\left(i - \sqrt{3}\right)I_{2} + \frac{1}{2}\left(i + \sqrt{3}\right)I_{3} - iI_{4},$$

$$J_{4} = -\frac{i\sqrt{2}}{2}I_{1} + I_{2} + I_{3} + I_{4},$$

$$(144)$$

introduces three additional square roots

$$s_3 = \sqrt{3}, \quad s_{-2} = i\sqrt{2}, \quad s_{-1} = i.$$
 (145)

From the definition of the basis *J* it follows that

$$J_4 = \rho_2(J_1) = \sigma_{-2}(J_1),$$

 $J_3 = \sigma_3(J_2).$ (146)

With the choice of the master integrals as in eq. (144) the square root $s_{-1} = i$ leads to the relation $J_3 = -\sigma_{-1}(J_2)$. However, it is sufficient to focus on the Galois symmetries in eq. (146). In the basis J, the differential equation is again in an ε -factorised form

$$dJ = \varepsilon AJ, \tag{147}$$

and is structured as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

$$(148)$$

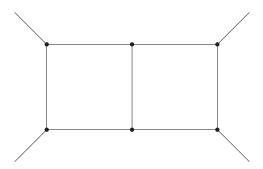


Figure 7: The Feynman diagram for the massless planar double-box integral.

We have the self-duality relations

$$a_{44} = a_{11}, \quad a_{33} = a_{22}, \quad a_{34} = a_{12}, \quad a_{24} = a_{13}, \quad a_{42} = a_{31}, \quad a_{43} = a_{21}.$$
 (149)

In addition we have the Galois symmetries

$$a_{13} = \sigma_3(a_{12}), \quad a_{31} = \sigma_3(a_{21}), \quad a_{42} = \rho_2(a_{12}) = \sigma_{-2}(a_{12}).$$
 (150)

We see that self-duality and Galois symmetries reduce the number of entries of A, which need to be known, to five (for example a_{11} , a_{22} , a_{14} , a_{23} and a_{12}), the remaining ones follow from symmetry.

3.5 The massless double-box integral

While we have seen that in sectors with more than one master integral we may have in addition to self-duality a Galois symmetry, this is not guaranteed. To illustrate this point, we discuss one of the simplest examples, the massless planar double-box integral [26] depicted in fig 7. We show that in this case we have a limit Galois symmetry as discussed in section 2.6. This example has eight master integrals, grouped into seven sectors. The first six sectors have one master integral each, the top sector consists of two master integrals. An ε -factorised form has been given in [8, 27]. We follow the notation of ref. [28]. The inverse propagators are given by

$$P_{1} = -(k_{1} - p_{1})^{2}, P_{2} = -(k_{1} - p_{1} - p_{2})^{2}, P_{3} = -k_{1}^{2},$$

$$P_{4} = -(k_{1} + k_{2})^{2}, P_{5} = -(k_{2} + p_{1} + p_{2})^{2}, P_{6} = -k_{2}^{2},$$

$$P_{7} = -(k_{2} + p_{1} + p_{2} + p_{3})^{2}, P_{8} = -(k_{1} - p_{1} - p_{3})^{2}, P_{9} = -(k_{2} + p_{1} + p_{3})^{2}. (151)$$

We denote the relevant kinematic variable by x = s/t, with s and t being the usual Mandelstam variables as defined in eq. (77). There are two differential one-forms,

$$\omega_0 = d \ln(x) \quad \text{and} \quad \omega_1 = d \ln(x+1). \tag{152}$$

Starting from the pre-canonical basis

$$K =$$
 (153)

 $\left(I_{001110000},I_{100100100},I_{011011000},I_{100111000},I_{111100100},I_{101110100},I_{111111100},I_{1111111(-1)0}\right)^{T}$ we obtain an ϵ -factorised basis through

$$I = U'UK, (154)$$

where U is given in eq. (6.232) of ref. [28] and U' is given by

$$U' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -3 & -1 & 1 & 0 & -2 & 1 & 0 \\ -2 & 2 & 0 & -2 & 0 & 4 & 0 & 1 \end{pmatrix}.$$
 (155)

The constant matrix U' maximises zeros in the last two rows. In the basis I we have a differential equation in ε -factorised form. It is easily checked that it is impossible to have self-duality and a Galois symmetry at the same time. There are no square roots which depend on the kinematic variable x. We may make self-duality manifest through a $GL(2,\mathbb{C})$ -transformation in the top sector. In this case we find that the required transformation is actually a $GL(2,\mathbb{Q})$ -transformation and does not introduce any square roots either. Hence, there is no Galois symmetry (in the usual sense) on top of self-duality. However, there is a limit Galois symmetry. The change of basis

$$J_7 = I_7 + \left(1 + \frac{s_1}{2}\right)I_8, \quad J_8 = I_7 + \left(1 - \frac{s_1}{2}\right)I_8$$
 (156)

with $s_1 = \sqrt{1}$ makes self-duality manifest. In addition we have a limit Galois symmetry

$$J_8 = \sigma_1(J_7).$$
 (157)

In practical terms we set $s_1 = 1$ in the end and the limit Galois symmetry is simply the substitution $s_1 \to -s_1$. In the basis J we have the ε -factorised differential equation

$$dJ = \varepsilon AJ \tag{158}$$

with

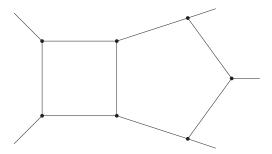


Figure 8: The Feynman diagram for the planar massless pentabox integral.

where entries with the same background colour are related by self-duality or a limit Galois symmetry. In detail we have

$$a_{77} = a_{88} = -2\omega_0 + \frac{1}{2}\omega_1,$$

 $a_{78} = -2(1+s_1)\omega_0 + \frac{1}{2}\omega_1, \quad a_{87} = -2(1-s_1)\omega_0 + \frac{1}{2}\omega_1.$ (160)

The entries a_{78} and a_{87} are related by

$$a_{87} = \sigma_1(a_{78}). (161)$$

The entries of the non-diagonal blocks are given by

$$a_{72} = a_{82} = -4\omega_0$$
, $a_{73} = a_{83} = 2\omega_0$, $a_{75} = a_{85} = -2\omega_1$, $a_{76} = a_{86} = -4\omega_1$. (162)

3.6 The planar pentabox integral

As our most involved example we discuss the planar massless pentabox integral shown in fig. 8. This is an example with 61 master integrals. An \(\varepsilon\)-factorised form has been given in ref. [29]. The motivation for this example is as follows: Up to now all examples where we had Galois symmetries on top of self-duality involved Feynman integrals with at least one massive internal propagator. On the other hand, the massless double-box integral discussed in section 3.5 had only a limit Galois symmetry on top of self-duality. The planar massless pentabox integral is a Feynman integral that has in almost all sectors (normal) Galois symmetries in addition to self-duality with the exception of two sectors, where we find limit Galois symmetries in addition self-duality. The second aspect which makes this example interesting is the fact that in two sectors we find one-parameter families of Galois symmetries.

There is one kinematic square root

$$r_1 = \sqrt{\Delta}, \qquad \Delta = \left(4i\varepsilon_{\mu\nu\rho\sigma}p_1^{\mu}p_2^{\nu}p_3^{\rho}p_4^{\sigma}\right)^2.$$
 (163)

The alphabet consists of 26 letters in the planar case. 21 letters are even with respect to $r_1 \rightarrow -r_1$, five letters are odd. The 61 master integrals group into 46 sectors. 34 sectors have one master integral, 9 sectors have two master integrals and 3 sectors have three master integrals. We find that it is possible to redefine the master integrals such that self-duality is manifest in all sectors. Furthermore it is possible to choose the master integrals such that we have Galois symmetries in all sectors except two. In these two sectors we do have limit Galois symmetries. The two sectors are the ones corresponding to the two double-box topologies with one external massive leg.

We denote by $I = (I_1, ..., I_{61})^T$ the basis of master integrals as given in ref. [29]. The symmetries are realised by the change of basis

$$J_{22} = I_{22} + \frac{\sqrt{3}}{6}I_{23}, \qquad J_{23} = I_{22} - \frac{\sqrt{3}}{6}I_{23},$$

$$J_{26} = I_{26} + \frac{\sqrt{3}}{6}I_{27}, \qquad J_{27} = I_{26} - \frac{\sqrt{3}}{6}I_{27},$$

$$J_{31} = I_{31} + \frac{\sqrt{3}}{6}I_{32}, \qquad J_{32} = I_{31} - \frac{\sqrt{3}}{6}I_{32},$$

$$J_{37} = I_{38} + \frac{2i\sqrt{3}}{3}I_{37}, \qquad J_{38} = I_{38} - \frac{2i\sqrt{3}}{3}I_{37},$$

$$J_{40} = I_{40} + \frac{i\sqrt{3}}{3}I_{41}, \qquad J_{41} = I_{40} - \frac{i\sqrt{3}}{3}I_{41},$$

$$J_{44} = I_{44} + \frac{i\sqrt{3}}{3}I_{45}, \qquad J_{45} = I_{44} - \frac{i\sqrt{3}}{3}I_{45},$$

$$J_{46} = I_{46} + \frac{i\sqrt{3}}{3}I_{47}, \qquad J_{47} = I_{46} - \frac{i\sqrt{3}}{3}I_{47},$$

$$J_{49} = I_{49} + \frac{(3-\lambda_1)}{2}I_{50} + 2\sqrt{\lambda_1}I_{51}, \qquad J_{51} = I_{49} + \frac{(3-\lambda_1)}{2}I_{50} - 2\sqrt{\lambda_1}I_{51},$$

$$J_{50} = i\sqrt{2}I_{49} + \frac{i\sqrt{2}(3+\lambda_1)}{2}I_{50},$$

$$J_{52} = I_{53} + \frac{1}{3}(1+s_1)I_{52}, \qquad J_{53} = I_{53} + \frac{1}{3}(1-s_1)I_{52},$$

$$J_{54} = I_{55} + \frac{1}{3}(1+s_1)I_{54}, \qquad J_{55} = I_{55} + \frac{1}{3}(1-s_1)I_{54},$$

$$J_{56} = I_{56} + \frac{(3-\lambda_2)}{2}I_{57} + 4\sqrt{\lambda_2}I_{58}, \qquad J_{57} = i\sqrt{2}I_{59} - 4\sqrt{\lambda_2}I_{58},$$

$$J_{57} = i\sqrt{2}I_{59} + \frac{i\sqrt{2}}{2}I_{57},$$

$$J_{60} = 5I_{59} + 3I_{61}. \qquad (164)$$

and $J_k = I_k$ for all other master integrals. Here we introduced the square roots

$$s_1 = \sqrt{1}, \quad s_2 = \sqrt{2}, \quad s_3 = \sqrt{3}, \quad s_{-3} = i\sqrt{3}, \quad s_{\lambda_1} = \sqrt{\lambda_1}, \quad s_{\lambda_2} = \sqrt{\lambda_2}.$$
 (165)

Sector	Galois symmetries
J_{22}, J_{23}	σ_3
J_{26}, J_{27}	σ_3
J_{31}, J_{32}	σ_3
J_{37}, J_{38}	$ ho_1, \sigma_{-3}$
J_{40}, J_{41}	$ ho_1, \sigma_{-3}$
J_{44}, J_{45}	$ ho_1, \sigma_{-3}$
J_{46}, J_{47}	$ ho_1, \sigma_{-3}$
J_{49}, J_{50}, J_{51}	$\rho_1,\sigma_{\lambda_1}$
J_{52}, J_{53}	σ_1
J_{54}, J_{55}	σ_1
J_{56}, J_{57}, J_{58}	$\rho_1,\sigma_{\lambda_2}$
J_{59}, J_{60}, J_{61}	σ_2

Table 5: Overview of the Galois symmetries of the sectors with more than one master integral.

The Galois symmetries associated with the sectors consisting of two or three master integrals are summarised in table 5. The sectors (J_{52},J_{53}) and (J_{54},J_{55}) have a limit Galois symmetry. Of particular interest are also the sectors (J_{49},J_{50},J_{51}) and (J_{56},J_{57},J_{58}) , where we have a parameterised Galois symmetry: For any value $\lambda_i \in \mathbb{Q}$ (with i=1,2) which is not a perfect square we have self-duality and a Galois symmetry.

We further remark that it is a matter of convention whether ρ_1 is considered to be a Galois symmetry of the top sector (J_{59}, J_{60}, J_{61}) . This is a sector with three master integrals and we required in this paper that any Galois symmetry acts trivially on the middle (second) master integral (see eq. (42)). In our example ρ_1 acts on J_{60} as

$$\rho_1(J_{60}) = -J_{60}. {166}$$

If one takes this additional minus sign into account one may view ρ_1 also as a Galois symmetry of the top sector.

3.7 Further examples

In addition we investigated several families of Feynman integrals with about O(30) master integrals each. First, we considered the full set of the two-loop mixed electroweak-QCD master integrals for the Drell-Yan process as discussed in refs. [22, 30, 31]. This family consists of 36 master integrals. The top-level diagrams are shown in fig. 9. This family contains the example discussed in section 3.1 as a sub-system. In addition, it has a sector with four master integrals.

Secondly, we investigated the family of the planar double-box integrals with one internal mass. Massive propagators are indicated by red lines in the left diagram of figure 10. The system consists of 29 master integrals. This integral was first analytically computed in ref. [32] via an ε -factorised differential equation.

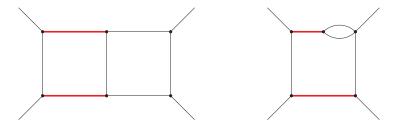


Figure 9: The top-level diagrams for the two-loop mixed electroweak-QCD master integrals for the Drell-Yan process. Black lines denote massless particles, red lines denote particles with a mass m.

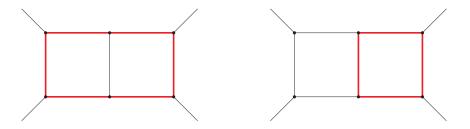


Figure 10: Two examples of planar double-box Feynman integrals with internal masses. Black lines denote massless particles, red lines denote particles with a mass m.

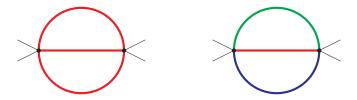


Figure 11: The Feynman diagram of the equal-mass sunrise integral (left) and the unequal-mass sunrise integral (right). Coloured lines denote massive particles and different colours correspond to different masses.

Thirdly, we studied a further planar double-box integral with one internal mass. The massive propagators are indicated by red lines in the right diagram of figure 10. This family differs from the previous one by the mass assignments for the internal propagators, and consists of 32 master integrals. The first analytic computation was presented in ref. [33], again via an ε -factorised differential equation.

These examples have been chosen as they provide sectors with up to four master integrals and several square roots. In all cases we were able to realise self-duality and Galois symmetries.

4 Beyond dlog-forms

It is well-known that there exist Feynman integrals whose differential equations involve differential one-forms beyond dlog-forms with algebraic arguments. A typical example are elliptic Feynman integrals, and more generally, Calabi–Yau Feynman integrals. In this section we discuss Galois symmetries for elliptic Feynman integrals. In the elliptic case there is an additional integer number, called the modular weight k. The known ϵ -factorised differential equations in the elliptic case have the property that each entry of the differential equation matrix A has a unique modular weight k. A Galois symmetry permutes roots of a polynomial equation, it does not change the modular weight. Hence, Galois symmetries cannot relate entries of different modular weight. For this reason, there is no Galois symmetry in the simplest elliptic Feynman integral, the equal-mass two-loop sunrise integral, shown in fig 11. However, there can be Galois symmetries relating entries of the same modular weight. As an example of a Galois symmetry in the elliptic case we discuss the unequal-mass sunrise integral, also shown in fig. 11 and a two-loop non-planar three-point function shown in fig. 12.

4.1 The equal-mass sunrise integral

The family of the equal-mass two-loop sunrise integral consists of two sectors: one sector with one master integral corresponding to the product of two one-loop tadpole integrals and a second sector with two master integrals corresponding to the sunrise topology. The differential equation

can be brought into an ε-factorised form and reads [34]

$$dI = \varepsilon AI, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega_2 & \omega_0 \\ \omega_3 & \omega_4 & \omega_2 \end{pmatrix}. \tag{167}$$

The sunrise integral depends on one kinematic variable, which we may take to be the modular parameter τ . In this variable we have

$$\omega_k = f_k(\tau) (2\pi i) d\tau, \tag{168}$$

where $f_k(\tau)$ is a modular form of modular weight k. The matrix A in eq. (167) clearly has the self-duality symmetry, which is the statement that

$$a_{22} = a_{33} = \omega_2.$$
 (169)

However, the modular form $f_0(\tau)$ appearing in $a_{23} = \omega_0$ is of modular weight 0, while the modular form $f_4(\tau)$ appearing in $a_{32} = \omega_4$ is of modular weight 4. There cannot be a Galois symmetry relating $f_0(\tau)$ to $f_4(\tau)$.

4.2 The unequal-mass sunrise integral

The situation is different for the unequal-mass sunrise integral. This is a system with seven master integrals. There are three sectors with one master integral each, corresponding to products of two one-loop tadpole integrals. In addition, there exists a sector with four master integrals corresponding to the sunrise topology. In the latter sector we may choose the master integrals such that two of them reduce to the master integrals of the equal-mass case in the equal-mass limit. As in the equal-mass case there is no Galois symmetry relating these two. The other two master integrals can be chosen such that they vanish in the equal-mass limit. More importantly, the entries in the differential equation related to these master integrals have the same modular weights. A redefinition of these will realise a Galois symmetry. In the following we will follow the notation of ref. [35]. We have an ε-factorised differential equation

$$dJ = \varepsilon AJ, \tag{170}$$

with a (7×7) -matrix A. The master integrals J are defined in eq. (78) of ref. [35]. The modular weights of the non-zero entries of A are

$$\begin{pmatrix}
2 & - & - & - & - & - & - \\
- & 2 & - & - & - & - & - \\
- & - & 2 & - & - & - & - \\
- & - & - & 2 & 1 & 1 & 0 \\
2 & 2 & 2 & 3 & 2 & 2 & 1 \\
2 & 2 & 2 & 3 & 2 & 2 & 1 \\
3 & 3 & 3 & 4 & 3 & 3 & 2
\end{pmatrix}$$
(171)

and we see that corresponding entries related to J_5 and J_6 have the same modular weights. Changing to a new basis

$$J_5' = \frac{1}{12}\sqrt{3}J_5 + \frac{i}{4}J_6, \qquad J_6' = \frac{1}{12}\sqrt{3}J_5 - \frac{i}{4}J_6, \tag{172}$$

and $J'_k = J_k$ for all other k will realise self-duality and the Galois symmetry

$$J_6' = \sigma_{-1}(J_5'). (173)$$

We have again introduced two square roots

$$s_3 = \sqrt{3}, \quad s_{-1} = i.$$
 (174)

In the new basis J' we have again an ε -factorised differential equation

$$dJ' = \varepsilon A'J', \quad A' = \begin{pmatrix} a'_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a'_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a'_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a'_{44} & a'_{45} & a'_{46} & a'_{47} \\ a'_{51} & a'_{52} & a'_{53} & a'_{54} & a'_{55} & a'_{56} & a'_{57} \\ a'_{61} & a'_{62} & a'_{63} & a'_{64} & a'_{65} & a'_{66} & a'_{67} \\ a'_{71} & a'_{72} & a'_{73} & a'_{74} & a'_{75} & a'_{76} & a'_{77} \end{pmatrix}. \tag{175}$$

The matrix A' satisfies now the self-duality relations

$$a'_{77} = a'_{44}, \quad a'_{66} = a'_{55}, \quad a'_{67} = a'_{45}, \quad a'_{57} = a'_{46}, \quad a'_{76} = a'_{54}, \quad a'_{75} = a'_{64}, \quad (176)$$

and in addition the Galois symmetries

$$a'_{65} = \sigma_{-1}(a'_{56}), \quad a'_{46} = \sigma_{-1}(a'_{45}), \quad a'_{64} = \sigma_{-1}(a'_{54}),$$
 (177)

and

$$a'_{6j} = \sigma_{-1}(a'_{5j}) \quad \text{for } j \in \{1, 2, 3\}.$$
 (178)

4.3 A non-planar elliptic three-point function

We consider the two-loop non-planar three-point function shown in fig. 12. This family consists of 11 master integrals, grouped into 8 sectors. There are three sectors with two master integrals each. Among the sectors with two master integrals there is the top sector, which is elliptic. An ε -factorised differential equation has been given in ref. [12], and we follow to a large extent the notation of this paper. As we always order the sectors from the simplest to the most complicated sector, we start from a basis I with

$$I_3 = M_8, I_4 = M_9, I_6 = M_5, I_7 = M_6, I_{10} = M_1, I_{11} = M_2,$$
 (179)

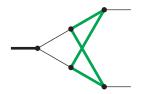


Figure 12: An example of a two-loop non-planar elliptic three-point function. Thin black lines denote massless particles, green lines denote particles with a mass m.

and $I_k = M_{12-k}$ for all others. The master integrals M_k are defined in ref. [12]. This example has two kinematic square roots

$$r_1 = \sqrt{1+4y}, \quad r_{-1} = \sqrt{1-4y},$$
 (180)

where y is the kinematic variable defined in ref. [12]. In order to realise self-duality and Galois symmetries, we introduce

$$s_3 = \sqrt{3}, \quad s_6 = \sqrt{6}$$
 (181)

and set

$$J_{3} = I_{3} + \frac{\sqrt{6}}{3}I_{4}, J_{4} = I_{3} - \frac{\sqrt{6}}{3}I_{4},$$

$$J_{6} = I_{7} + \frac{\sqrt{3}}{3}I_{6}, J_{7} = I_{7} - \frac{\sqrt{3}}{3}I_{6}.$$
(182)

We have the Galois symmetries

$$J_4 = \rho_1(J_3) = \sigma_6(J_3), \quad J_7 = \rho_{-1}(J_6) = \sigma_3(J_6).$$
 (183)

There is no Galois symmetry for the elliptic sector consisting of the master integrals J_{10} and J_{11} (for the same reasons as discussed in section 4.1). In the basis J we have again an ε -factorised differential equation

$$dJ = \varepsilon AJ, \tag{184}$$

where the (11×11) -matrix A has the structure (we use again hexadecimal indices)

Entries with the same background colour are related by a symmetry. Note that there is no Galois symmetry relating a_{AB} to a_{BA} . The former entry has modular weight 0, the latter modular weight 4.

5 Conclusions

In this paper we studied the structure of the differential equation for families of Feynman integrals. We showed that often the connection matrix A has additional symmetries, which can be realised by a redefinition of the master integrals through a constant $GL(n, \mathbb{C})$ -transformation. The symmetries we studied were self-duality and Galois symmetry. Self-duality is the statement that blocks on the diagonal are symmetric with respect to the anti-diagonal. In all examples we presented we were able to find a transformation which achieves self-duality.

In addition to self-duality there can be Galois symmetries present. Galois symmetry relates two master integrals through the action of a Galois group: $I_2 = \sigma(I_1)$. Galois symmetries can be expected in sectors where the definition of the master integrals for an ε-factorised differential equation involves square roots. Surprisingly, also sectors not related to any square root with dependence on the kinematic variables may exhibit Galois symmetries. The requirement of selfduality can introduce constant square roots like $\sqrt{3}$ and the Galois symmetry in the latter case is the conjugation $\sqrt{3} \rightarrow -\sqrt{3}$. Galois symmetries may or may not exist on top of self-duality. We presented many examples with Galois symmetries, but we also pointed out two examples, where there is no Galois symmetry on top of self-duality: These were the examples of the two-loop planar massless double-box integral and the two-loop equal-mass sunrise integral. In the former case we showed that we still have a limit Galois symmetry on top of self-duality. In the latter case we do not expect Galois symmetries, as Galois symmetries cannot relate quantities of different modular weight. To the other extreme, we also find examples in which a Galois symmetry can be realised in a number of different ways. Moreover, there exist cases with symmetries involving arbitrary roots, including perfect squares. Thus the presence of Galois symmetry is not necessarily linked with a specific or even any field extension (see also the discussion in section 2.5).

Clearly, self-duality and Galois symmetries give only non-trivial relations in sectors with two or more master integrals and therefore only affect more complicated Feynman integrals. This might be an explanation why these symmetries have not been noticed up to now. In this paper we presented strong evidence for the ubiquity of self-duality and Galois symmetries in Feynman integrals. It would be interesting to understand the exact conditions under which these symmetries can be realised, and to which extent they are unique. This is left for future work.

Note added

While this paper was under review, ref. [36] appeared on the arXiv, providing an explanation of the self-duality symmetry we observed in terms of twisted cohomology on the maximal cut.

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